THE UNIVERSITY OF
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DOA Estimation for Automotive Radar

by

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Topic Title: **DOA Estimation for Automotive Radar**

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A. **Problem statement**

Direction of Arrival (DOA) estimation is an important branch of array signal processing. DOA estimation aims to determine the location of one or multiple signals in a space at the same time. More specifically, it aims to find the direction (angle) of signals arrived to the array sensors. The automotive radar is widely used nowadays as a driving assistance system device and currently attracts much attention and has been widely concerned. The automotive radar depends on DOA estimation technique to determine the direction of the target vehicles. Therefore it is meaningful to investigate the different methods of DOA estimation and evaluate how each method performs on automotive radar system.

B. **Objective**

Do literature survey at first. Understand the theory of DOA estimation. Understand the mathematical signal model expressions. Then search different typical methods that used, such as conventional FFT-based approach, MUSIC algorithm, Capon beamforming, Maximum Likelihood method. Finally based on one specific existing method, investigate an improved algorithm which could significantly reduce the computational complexity but get a good performance of accuracy.

C. **My solution**

Capon beamforming method simulation implementation
Conventional FFT-based estimation simulation implementation
MUSIC algorithm simulation implementation
Proposed algorithm simulation implementation

D. **Contributions** (at most one per line, most important first)

Compare the proposed algorithm with conventional FFT in terms of *SNR* vs. RMSE
Compare the proposed algorithm with conventional FFT in terms of *M* vs. RMSE
Compare computational times reduced in the proposed algorithm
Show the pros and cons of conventional FFT-based method
Show the pros and cons of MUSIC algorithm
Show the pros and cons of Capon beamforming

E. **Suggestions for future work**

Correctly get the comparison results in terms of multiple source signals case.
If possible, do hardware implementation by using microprocessor.

While I may have benefited from discussion with other people, I certify that this report is entirely my own work, except where appropriately documented acknowledgements are included.

Signature: ____________谷博文_______________ Date: _26_ / _10_ / _2016_

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22	[16] Elias Aboutanios, 2016
15	[11] G. Heinzel, 2002
16	[3] Kundu D, 1996
12	[8] C. K. E. Lau, 2004
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Abstract

Array signal processing is an important branch of signal processing, and has a rapid development in recent years [1, 2]. One of the main aspects of array signal processing is to investigate the spatial spectrum estimation [3, 4]. If spatial spectrum is known, the Direction of Arrival (DOA) of the signal could be known. So, generally spatial spectrum estimation is also known as DOA estimation [3]. DOA estimation aims to determine the location of one or multiple signals in a space at the same time. More specifically, it aims to find the direction (angle) of signals arrived to the array sensors [3]. The automotive radar is widely used nowadays as a driving assistance system device and currently attracts much attention and has been widely concerned [5]. The automotive radar depends on DOA estimation technique to determine the direction of the target vehicles. Therefore it is meaningful to investigate the different methods of DOA estimation and evaluate how each method performs on automotive radar system.

Keywords: DOA estimation, automotive radar, Conventional FFT-based method, MUSIC algorithm, a proposed algorithm.

Abbreviations

DOA	Direction of Arrival
ULA	Uniform Linear Array
FFT	Fast Fourier Transform
BF	Beamforming
DBF	Digital Beamforming
CRB	Cramer-Rao Bound
CB	Capon Beamforming
ML	Maximum Likelihood
MUSIC	Multiple Signal Classification
SNR	Signal to Noise Ratio
RMSE	Root Mean Square Error

Content

Chapter 1

Introduction

My report is organized as follows. In section 1, an introduction of DOA estimation and automotive radar is given. Section 2 is the literature survey which describes the aspects of beamforming technique [6], automotive radar model [7], signal model [8], CRB [9], Capon beamforming method [6, 10], FFT-based linear spectrum estimation [11], ML estimation and MUSIC algorithm [3, 12]. In section 3, a proposed algorithm based on conventional FFT method is demonstrated, in terms of improving the performance and reducing computational complexity. In section 4, a simulation result about my work is shown. Finally, the reference is given.

1.1 DOA estimation

Array signal processing is an important branch of Signal Processing, and has a rapid development in recent years. There are lots of applications of this technique which involves the fields of Radar, Sonar, Astronomy, Seismology, Satellite Navigation, Medicine and so on.

The aim of array signal processing is to process the signal received by the array sensors, enhance the strength of the desired signal and restrain the strength of the undesired signal and noise, finally extract the useful information of the desired signal. By comparison with traditional single sensor, array sensor has the advantages of flexible beam direction control, high signal gain, good performance of high resolution [3].

One of the main aspects of array signal processing is to investigate the spatial spectrum estimation. Spatial spectrum estimation focuses on how array signal system could get the spatial signal parameters as accurate as possible, and its main task is to estimate the location of the source signal and spatial parameters. This is also the main task in the field of Radar, Sonar and Telecommunication [1, 2].

A spatial spectrum shows signal energy distribution in different directions in space. Therefore, if spatial spectrum is known, the Direction of Arrival (DOA) of the signal could be known. So, generally spatial spectrum estimation is also known as DOA estimation. In some references, DOA estimation is also called bearing estimation, angle estimation or direction finding. Actually, they are the synonyms [3].

As an important and popular topic in array signal processing, DOA estimation aims to determine the location of one or multiple signals in a space at the same time. More specifically, it aims to find the direction (angle) of signal arrived to the array antenna. In practice, the signal is corrupted by the added noise. Therefore, it is required to extract the direction information of desired signal from noise.

In my report, there are several methods listed, for instance, Capon beamforming method, conventional FFT-based estimation, Maximum Likelihood estimation, Multiple Signal Classification (MUSIC) algorithm. It is hard to simply say which method is absolutely effective by comparison with others, since each method could be effective under a specific condition and dose not perform well when the condition changes. Also, there are many factors need to be considered when evaluating a specific method. For example, it is important to make a trade-off between the computational complexity and required performance depending on specific situation.

1.2 Automotive Radar

The automotive radar is widely used nowadays as a driving assistance system device. Based on acquiring the information (relative velocity, relative range, direction of arrival) from the vehicles in front and analyzing the probability of collision, the driver can receive suggestions from the alarm system and make any reaction in time. Or even the case, if the system finds that the collision is definitely to be happened, it will automatically interfere the action to brake the car at once to make sure the security. Therefore, the automotive radar currently attracts much attention and has been widely concerned [5].
Automotive Radar largely depends on DOA estimation to get correct information of direction of the targets. A good DOA estimation result could provide sufficiently accurate information in real time. So both accuracy and processing speed should be taken as important factors when evaluating each DOA estimation method.

1.3 Pre-requisite knowledge

Random Process [14]
Mean (expectation)

$$\mu_{x[n]} = E\big[x[n]\big] = \int_{-\infty}^{\infty} \alpha \, p_{x[n]}(\alpha) d\alpha$$

Mean square value

$$E[x[n]^2] = \int_{-\infty}^{\infty} \alpha^2 \, p_{x[n]}(\alpha) d\alpha$$

Variance

$$\sigma_{x[n]}^2 = E\left[\left(x[n] - \mu_{x[n]}\right)^2\right] = \int_{-\infty}^{\infty} (\alpha - \mu_{x[n]})^2 \, p_{x[n]}(\alpha) d\alpha$$

Autocorrelation

$$\varphi_{xx}[n,m] = E[x[n]x[m]^*] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha\beta^* p_{x[n],x[m]}(\alpha,\beta) d\alpha d\beta$$

Auto covariance

$$\gamma_{xx}[n,m] = E\left[\left(x[n] - \mu_{x[n]}\right)\left(x[m] - \mu_{x[m]}\right)^*\right]$$

$$\gamma_{xx}[n,m] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\alpha - \mu_{x_{[n]}})(\beta - \mu_{x_{[m]}})^* p_{x[n],x[m]}(\alpha,\beta)d\alpha d\beta$$

Cross correlation

$$\varphi_{xy}[n,m] = E[x[n]y[m]^*] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha\beta^* p_{x[n],y[m]}(\alpha,\beta)d\alpha d\beta$$

Cross covariance

$$\gamma_{xy}[n,m] = E\left[\left(x[n] - \mu_{x[n]}\right)\left(y[m] - \mu_{y[m]}\right)^*\right]$$

$$\gamma_{xy}[n,m] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\alpha - \mu_{x_{[n]}})(\beta - \mu_{y_{[m]}})^* p_{x[n],y[m]}(\alpha,\beta)d\alpha d\beta$$

If $\boldsymbol{v}$ refers to a *n-th* order random vector, $\boldsymbol{v} = [X_1, X_1, \dots, X_n]$

$$c_{ij} = Cov(X_i, X_j) = E\left[\left(X_i - E(X_i)\right)\left(X_j - E(X_j)\right)\right] \quad i,j = 1,2,\dots,n$$

$\boldsymbol{C} = \{c_{ij}\}$ is defined as the covariance matrix of vector $\boldsymbol{v}$.

Covariance matrix $\boldsymbol{C}$ has the characteristics that it is positive definite and symmetric, which means $\boldsymbol{C}^T = \boldsymbol{C}$.

Hermite Matrix [15]

A Hermite Matrix (also called self-adjoint matrix) is a square matrix with complex entries that is equal to its own conjugate transpose[wiki]. The element in the *i-th* row and *j-th* column is equal to the complex conjugate of the element in the *j-th* row and *i-th* column, which can be expressed as:

$$\overline{a_{ij}} = a_{ji} \ \ (i,j = 1,2,\dots,n)$$

From the above equation, we can see that diagonal elements of Hermite Matrix are all real values.

Assume $\boldsymbol{A^T}$ and $\overline{\boldsymbol{A}}$ are transpose and conjugate matrix of matrix $\boldsymbol{A}$, respectively. The sufficient and necessary condition for matrix $\boldsymbol{A} = [a_{ij}]$ to be a Hermite Matrix is：

$$\boldsymbol{A^T} = \overline{\boldsymbol{A}}$$

There are several characteristics of Hermite Matrix:

(1) If matrix $\boldsymbol{A}$ is Hermite Matrix, then $|\boldsymbol{A}|$ is real valued.

(2) If matrix $\boldsymbol{A}$ is Hermite Matrix, k is any real valued number, $k\boldsymbol{A}$ is still Hermite Matrix.

(3) If matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$ are both Hermite Matrices, $\boldsymbol{A} + \boldsymbol{B}$ is Hermite Matrix.

(4) If matrix $\boldsymbol{A}$ is Hermite Matrix, $\boldsymbol{A^T}$, $\overline{\boldsymbol{A}}$, $\boldsymbol{A^H}$ are all Hermite Matrices. If $\boldsymbol{A}$ is invertible, $\boldsymbol{A}^{-1}$ is also Hermite Matrix.

Chapter 2

Literature Survey

2.1 Beamforming Technique

Beamforming technique is to reconstruct the source signal from the array sensors. This could be done by considering two important aspects.

(1) The first aspect is to increase the contribution of the expected source signal.

(2) The other aspect is to restrain the interference signal, such as noise.

The basic idea of beamforming is multiplying each sensor by a different weighting vector and thus steer the beam of the array sensors to a specific direction. DOA estimation is to get the direction of the maximum power of the desired signal.

Though the direction of the array sensors could reach 360-degree angles, once add the weighting on each array sensor and get sum of every sensor, the gain of the received signal could be adjusted and focused on one specific direction only. It is just like formulating a "beam". So this is the physical meaning of beamforming technology.

The advantage of using beamforming technology in DOA estimation is to significantly increase signal to noise ratio (SNR) and effectively improve the quality of received signal information.

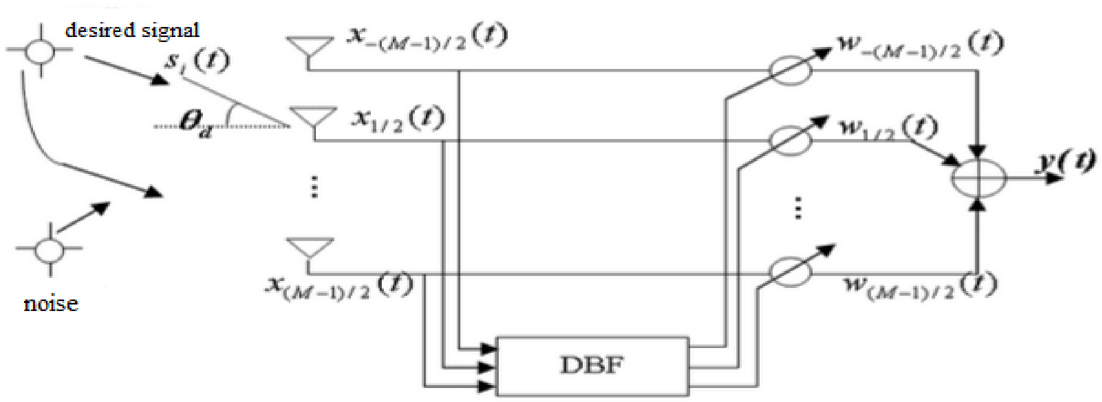

Figure2.1.1 Beamforming technic

The optimal weighting vector $\boldsymbol{w}$ is determined by the array sensor steering vector $\boldsymbol{s}(\phi_m)$. Before calculating the optimal weighting vector, it is necessary to know the geometric configuration of the array sensors. Then next step is to do DOA estimation of the desired signal.

2.2 Automotive Radar Model

Figure2.2.1 shows the model that how DOA estimation works for automotive radar. An automotive radar system with an array of antennas can be used to determine the DOA of signals which corresponding to different vehicles in front.

The configuration of the radar antenna is Uniform Linear Array (ULA). Though it is not always the case that requires the geometric of the radar antenna to be ULA (for some specific DOA estimation method, such as ESPRIT), here we just assume that the array antenna used in automotive is ULA.

By using beamforming technology, we can adjust the weighting coefficients of each antenna, and excited the signal in the specific direction. Also, the array of antenna could receive the steering vector which is related to the signal. Based on the relationship between the excitation and received steering vector, by doing Fourier Transform, we can estimate the spectrum and get the direction information ϕ.

Here, we define d as the equivalent space between each array of antennas. The number of antenna is defined as M.

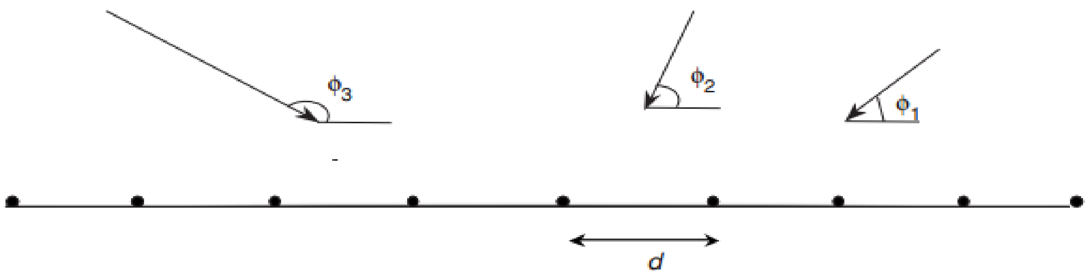

Figure2.2.1 ULA antenna model and steering vector with different direction

2.3 Signal Model

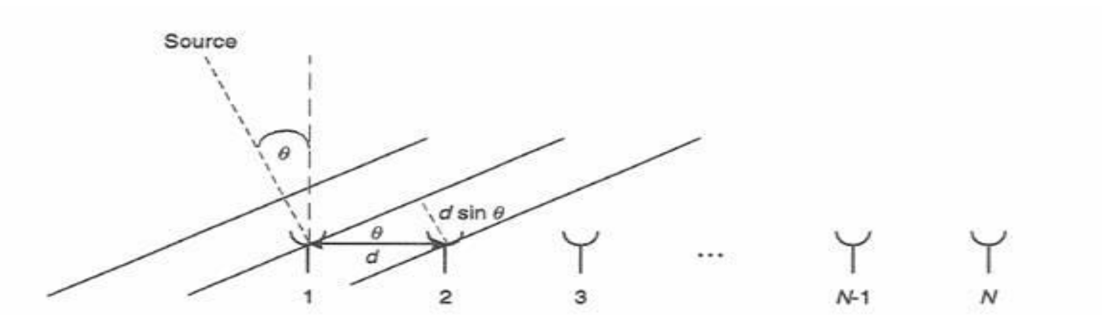

Figure2.3.1 DOA signal model

There are several assumptions made when doing DOA estimation:

(1) It is assumed that all the source signals are single points, there is no spread angle when looking back from the antennas to the source signal. The direction of the signal is unique.

(2) Each of the source signals is uncorrelated with other source signals. The source signal is considered to be narrow band, and has the same center frequency

$$\omega_0 = 2\pi f_0$$

(3) The equivalent space d between each antenna is no more than half length of the highest frequency signal.
(4) The array of antennas is in the far field from the signal sources. Therefore, the signal received by the antennas could be seen as parallel.
(5) There is added noise in each array of antennas. The noises are uncorrelated with each other, also uncorrelated with each source signals.
(6) The characteristic of each antenna is exactly the same.

There exists a difference of received wave-path between each antenna, and could be expressed as:

$$\tau = \frac{dcos\phi_i}{c}$$

Then the difference of phase between each antenna is:

$$\theta = e^{-j\omega\tau} = e^{-j\omega\frac{dcos\phi_i}{c}} = e^{-j2\pi\frac{dcos\phi_i}{\lambda f_0}f}$$

For narrow band signal as assumed, $f = f_0$, so

$$\theta = e^{-j2\pi\frac{dcos\phi_i}{\lambda}}$$

Consequently, if phase difference is known, according to the equation above, the DOA of the signal ϕ_i is known.

Generally, the received signal model can be defined as followed:

$$\mathbf{x} = \sum_{i=1}^{I} \alpha_i \boldsymbol{s}(\phi_i) + \boldsymbol{n}$$

And

$$\boldsymbol{s}(\phi_i) = \frac{1}{\sqrt{M}}[1, e^{j\phi_i}, \dots, e^{j(M-1)\phi_i}]$$

Here i stands for the *i-th* received signal and the total number of signals is *I*. α_i and ϕ_i stand for the amplitude and direction parameters of *i-th* signal, respectively. $\boldsymbol{s}(\phi_i)$ refers to the *i-th* steering vector. $\boldsymbol{n}$ refers to the noise vector, which is a zero-mean Gaussian with covariance $\sigma^2\boldsymbol{I}$.

Usually, we are interested in single ($I = 1$) or double target ($I = 2$) situations, so the signal model could be expressed as:

$$I = \mathbf{1}: \ \mathbf{x} = \alpha_1 \boldsymbol{s}(\phi_1) + \boldsymbol{n}$$

$$I = \mathbf{2}: \ \mathbf{x} = \alpha_1 \boldsymbol{s}(\phi_1) + \alpha_2 \boldsymbol{s}(\phi_2) + \boldsymbol{n}$$

2.4 Cramer-Rao Bound

Before looking at different methods, it is necessary to know the Cramer-Rao Bound (CRB). No matter which method is used, the minimum variance of any unbiased estimation method could not be less than CRB.

For a steering array, we can express the elements as:

$$s(\phi) = [z^{-\frac{N-1}{2}}, z^{-\frac{N-3}{2}}, \ldots, z^{-1}, 1, z, z^{\frac{N-3}{2}}, z^{\frac{N-1}{2}}]$$

Where

$$\mathrm{z} = e^{jkdcos\phi}$$

By constructing function and doing first and second order differentiation, finally got:

$$var(\phi) \geq \frac{6\sigma^2}{|\alpha|^2 N(N^2-1)(kd)^2 sin^2\phi}$$

It can be seen from the equation that CRB sets the best possible estimation. Also, with the increasing of the SNR, $\frac{\sigma^2}{|\alpha|^2}$ decreases, leading to a reduced CRB.

2.5 Capon Beamforming

The conventional (Bartlett) beamforming method suffers a significant problem that if there are multiple signal sources from different directions, especially the case the sources are closely, the resolution of the spectrum could be rather low, it could even fail to detect the right direction of each signal.

Compared with conventional beamforming, the Capon beamforming method provides better performance on resolution. The algorithm of Capon Beamforming is simple, it aims to minimize the output power of the array signal. By doing this, the contribution of undesired signal (noise) could be minimized as well. At the same time, it maintains the power in the direction of desired source signal, which means to keep the gain in the direction of desired source signal as a constant (The gain value is generally to be selected as 1). Capon beamforming algorithm could be expressed as followed:

$$\begin{cases} \min E[|y(k)|^2] = \min \boldsymbol{w}^T \boldsymbol{R}_{\mathbf{xx}} \boldsymbol{w} \\ st: \ \boldsymbol{w}^T \boldsymbol{s}(\phi_i) = 1 \end{cases}$$

This is a constrained optimization question that could be transferred to a non-constrained question by using Lagrange operator:

$$\boldsymbol{w} = \frac{\boldsymbol{R}_{\mathbf{xx}}{}^{-1} \boldsymbol{s}(\phi_i)}{\boldsymbol{s}^H(\phi_i) \boldsymbol{R}_{\mathbf{xx}}{}^{-1} \boldsymbol{s}(\phi_i)}$$

By using Capon beamforming method, the function of DOA could be expressed by Capon spatial spectrum:

$$P_{Capon}(\phi_i) = \frac{\mathbf{1}}{\boldsymbol{s}^H(\phi_i) \boldsymbol{R}_{\mathbf{xx}}{}^{-1} \boldsymbol{s}(\phi_i)}$$

When get all values and get the Capon spectrum, the DOA of each signal could be known as the peak value appears in the spectrum.

Here, $|y(k)|^2$ is the output power. $\boldsymbol{w}$ is the weighting vector as mentioned in **2.1**. $\boldsymbol{R}_{\mathbf{xx}}$ is the correlation matrix of the received signal.

However, there are still shortcomings in Capon beamforming method. Firstly, in terms of computational complexity, Capon beamforming requires to calculate the inverse of correlation matrix, which might be quite computationally complex when the correlation matrix is large.

Additionally, Capon beamforming is lack of the ability to distinguish the desired signal from other correlated undesired signal. This is because when minimizing the output power, it required the correlated relationship between desired signal and undesired signal. Finally, if there are too close source signals, Capon beamforming still could not resolve them and some high resolution methods might be required.

2.6 Conventional FFT-based DOA Estimation

Conventional FFT-based estimation is the most commonly known method in DOA estimation. It has the advantage of really simple algorithm and easy to implement.

Recall the signal model in **2.3**:

$$\mathbf{x} = \sum_{i=1}^{I} \alpha_i \boldsymbol{s}(\phi_i) + \boldsymbol{n}$$

Once the received signal vector **x** is obtained, do Fast Fourier Transform of received signal by a proper number of zero-tapping at first. This is expressed as:

$$X[n] = \sum_{k=0}^{L-1} \mathbf{x}[k] e^{-j2\pi k \frac{n}{L}}$$

Where L is the total FFT points which equal:

$$L = z \times M, \quad z = 1,2,\cdots$$

Where z is the integer multiple of zero-tapping.

Once got the FFT of signal vector **x**, further calculate the power spectrum of signal vector **x:**

$$P[n] = conj(X[n]) \times X[n] = |X[n]|^2$$

By searching the peak value appear in the spectrum, the desired source signals' DOA could be found.

However, there is a significant problem in this approach, which is undesirable sidelobe leakage in the spectrum. There is countermeasure to alleviate this problem:

For reducing the size of sidelobe leakage, a windowing function can be added:

$$X_W[n] = \sum_{k=0}^{L-1} w[k] \boldsymbol{x}[k] e^{-j2\pi k \frac{n}{L}}$$

And the new spectrum could be calculated:

$$P_W[n] = conj(X_W[n]) \times X_W[n] = |X_W[n]|^2$$

Here $w[k]$ is the windowing function.

Generally, $w[k]$ can be selected as rectangular window w^R[k] or hamming window w^H[k]:

$$w^R[k] = \begin{cases} 1 & |k| < M \\ 0 & |k| \geq M \end{cases}$$

and

$$w^H[k] = \begin{cases} 0.54 + 0.46cos\frac{\pi}{M}k & |k| < M \\ 0 & |k| \geq M \end{cases}$$

It can be seen that the new power spectrum $P_W[n]$ is the convolution of windowing function $w[k]$ and previous power spectrum $P[n]$.
For conventional FFT-based method, adding the windowing function can effectively reduce the size of sidelobes and increase the size of mainlobe. However, increasing mainlobe means to reduce the resolution in the spectrum. This is a contradiction inside FFT-based linear spectrum estimation method. Moreover, if there are multiple closely targets, FFT-based linear spectrum estimation method will lose the ability to resolve the closely signal due to spectral leakage. Spectrum leakage problem is always a significant problem by comparison with other more effective methods.
Besides, in terms of computational times, as doing zero-tapping with a sufficiently large multiple requires to increase computational times accordingly, this method could get a reasonable accuracy with efficient computational cost.

2.7 Maximum Likelihood Estimation

The Maximum Likelihood method is to maximize the likelihood that the received signal coming from the particular direction. It is possible to implement this method to solve single or even multiple target problems. Here just take single target as example.
The Maximum Likelihood Estimator is given as:

$$\hat{\phi}, \hat{\alpha} = \max_{\phi,\alpha}[f_{\mathbf{x}/\phi,\alpha}(\mathbf{x})]$$

Where $f_{x/\phi,\alpha}(x)$ is the pdf of the data vector x with the given parameters α, ϕ. Assuming that the noise vector is complex Gaussian,

$$f_{\mathbf{x}/\phi,\alpha}(\mathbf{x}) = \frac{1}{\pi^N det(\boldsymbol{R}_n)} e^{-(\mathbf{x}-\alpha\boldsymbol{s})^H \boldsymbol{R}_n{}^{-1}(\mathbf{x}-\alpha\boldsymbol{s})}$$

So equivalently, we need to get

$$\hat{\phi}, \hat{\alpha} = \min_{\phi,\alpha}\left[(\mathbf{x}-\alpha\mathbf{s})^H \boldsymbol{R}_n{}^{-1}(\mathbf{x}-\alpha\mathbf{s})\right]$$

Using differentiation, finally we can get

$$\hat{\alpha} = \frac{\boldsymbol{s}^H \boldsymbol{R}_n{}^{-1}\mathbf{x}}{\boldsymbol{s}^H \boldsymbol{R}_n{}^{-1}\boldsymbol{s}}$$

Using this value of $\hat{\alpha}$, we can get

$$\hat{\phi} = \max_{\phi}\left[\frac{\left|\boldsymbol{s}^H \boldsymbol{R}_n{}^{-1}\mathbf{x}\right|^2}{\boldsymbol{s}^H \boldsymbol{R}_n{}^{-1}\boldsymbol{s}}\right]$$

The DOA estimate is the point where this function takes its maximum.

However, for more than one target case, such as two targets, the BF technology could cause undesirable spectrum leakage. Particularly, for multiple target case, there is a method called RELAX algorithm which could eliminate the leakage.

RELAX Algorithm for DOA Estimation [13]
For the case with two targets, using RELAX algorithm aims to minimize the function:

$$\|\mathbf{x} - \alpha_1 \boldsymbol{s}(\phi_1) - \alpha_2 \boldsymbol{s}(\phi_2)\|^2$$

The minimization function above can be simplified to:

$$\hat{\phi}_i = \mathrm{argmax}\left|\boldsymbol{s}(\phi_{\mathrm{i}})^{\mathrm{H}}\mathbf{x}_{\mathrm{i}}\right|^2$$

$$\hat{\alpha}_i = \boldsymbol{s}(\phi_{\mathrm{i}})^{H}\mathbf{x}_{\mathrm{i}} \quad \mathrm{i} = 1{,}2$$

The RELAX algorithm, for the case of two targets, is summarized as;

(1) Assume a single target present, estimate parameters $\hat{\phi}_1$ & $\hat{\alpha}_1$ from $\mathbf{x}$.
(2) Assume two targets present, compute

$$\mathbf{x}_2 = \mathbf{x} - \hat{\alpha}_1 \boldsymbol{s}(\hat{\phi}_1)$$

(3) Using the previous estimates, get the estimated value of $\hat{\phi}_2$ & $\hat{\alpha}_2$.
(4) Re compute the function

$$\mathbf{x}_1 = \mathbf{x} - \hat{\alpha}_2 \boldsymbol{s}(\hat{\phi}_2)$$

(5) get the estimated value of $\hat{\phi}_1$ & $\hat{\alpha}_1$ from $\mathbf{x}_1$.
(6) Do the iteration for several loops, when the difference between two iterations are smaller than the expected threshold ϵ, stop the iteration and get the results. Otherwise, continue the steps above.

It can be seen that the RELAX algorithm can be effectively used to solve multi-targets signal problem. The advantage of this method is it could eliminate the spectrum leakage. However, because of a iterative implementation, the threshold needs to be determined suitable, because a smaller convergence requires more computation, while lower computational cost reduce the accuracy of the result.

2.8 Multiple Signal Classification (MUSIC algorithm)

Music algorithm is a popular method which belonging to spatial spectrum estimation. MUSIC algorithm aims to decompose the eigenvectors of the covariance matrix of the array signal, and get the related signal subspace and noise subspace, which are orthogonal. Base on this orthogonal characteristic, the spectrum function could be constructed, and the DOA of the signal could be got by searching the peak value in the spectrum.

There are many advantages of MUSIC algorithm:

(1) It has the ability to detect the DOA of multiple signals at the same time.
(2) High accuracy and resolution.
(3) When using high speed processing technology, it is possible to process the signal in real-time.

Recall the FFT-based method, it suffers the significant problem of spectrum leakage. However, MUSIC algorithm dose not suffer this problem and provide high resolution and accuracy.

Recall the signal model in **2.3**,

$$\mathbf{x} = \sum_{i=1}^{I} \alpha_i \boldsymbol{s}(\phi_i) + \boldsymbol{n}$$

Using definition of matrix, the expression could be simplified as:

$$\boldsymbol{X} = \boldsymbol{S\alpha} + \boldsymbol{N}$$

Here

$$\boldsymbol{X} = [\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_N]^T$$

$$\boldsymbol{S} = [\boldsymbol{s}(\phi_1), \boldsymbol{s}(\phi_2), \dots, \boldsymbol{s}(\phi_M)]$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_I} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(M-1)\phi_1} & e^{j(M-1)\phi_2} & \cdots & e^{j(M-1)\phi_I} \end{bmatrix}$$

$$\boldsymbol{\alpha} = [\alpha_1,\ \alpha_2, \dots,\ \alpha_I]^T$$

$$\boldsymbol{N} = [n_1, n_2, \dots, n_M]^T$$

Here $\boldsymbol{S}$ is a $M \times I$ matrix.

The signal covariance matrix of $\mathbf{x}$ can be written as:

$$\boldsymbol{R} = E[\boldsymbol{XX}^H]$$

Here assuming the different signals are uncorrelated, then

$$\begin{aligned} \boldsymbol{R} = E[\boldsymbol{XX}^H] &= E[\,(\boldsymbol{S\alpha} + \boldsymbol{N})(\boldsymbol{S\alpha} + \boldsymbol{N})^H] \\ &= \boldsymbol{S}E[\boldsymbol{\alpha\alpha}^H]\boldsymbol{S}^H + E[\boldsymbol{NN}^H] \\ &= \boldsymbol{SR}_s\boldsymbol{S}^H + \sigma^2\boldsymbol{I} \\ &= \boldsymbol{SR}_s\boldsymbol{S}^H + \boldsymbol{R}_n \end{aligned}$$

Here

$$\boldsymbol{R}_s = [\boldsymbol{\alpha\alpha}^H]$$

is the correlation matrix of signal, it is a diagonal matrix and can be expressed as:

$$\boldsymbol{R}_s = \begin{bmatrix} E[|\alpha_1|^2] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E[|\alpha_I|^2] \end{bmatrix}$$

And

$$\boldsymbol{R}_n = \sigma^2\boldsymbol{I}$$

Is the correlation matrix of noise.

In practice, the matrix $\boldsymbol{R}$ is unknown, it can only estimate the covariance matrix $\widehat{\boldsymbol{R}}$ from the received signal:

$$\widehat{\boldsymbol{R}} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{x}_k\, \mathbf{x}_k{}^H$$

$\widehat{\boldsymbol{R}}$ is the maximum likelihood estimation of $\boldsymbol{R}$, when sampling numbers $K \to \infty$, $\widehat{\boldsymbol{R}}$ is identical to $\boldsymbol{R}$. Practically, due to the limitation of sampling numbers, it will lead to estimation deviation.

Considering ideal situation at first, which means there is no noise added. Do eigendecomposition of signal covariance matrix:

$$\boldsymbol{R} = \boldsymbol{SR}_s\boldsymbol{S}^H$$

As ULA, $\boldsymbol{S}$ has the characteristic that:

$$\phi_i \neq \phi_j \quad i \neq j$$

So each column of matrix $\boldsymbol{S}$ is independent. Because it is assumed the signals are uncorrelated, the rank of matrix $\boldsymbol{R}_s$

$$Rank(\boldsymbol{R}_s) = M$$

So

$$Rank(\boldsymbol{R}) = Rank(\boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H) = M$$

Since

$$\boldsymbol{R} = E[\boldsymbol{X}\boldsymbol{X}^H]$$

So

$$\boldsymbol{R}^H = \boldsymbol{R}$$

Which means $\boldsymbol{R}$ is a Hermite matrix and all its eigenvalues are real valued. Also, matrix $\boldsymbol{R}_s$ is positive definite, therefore matrix $\boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H$ is positive semidefinite, it has I positive eigenvalues and $M - I$ zero eigen values.

Now considering the situation with noise added:

$$\boldsymbol{R} = \boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H + \sigma^2\boldsymbol{I}$$

Because $\sigma^2 > 0$, and $\boldsymbol{R}$ is full rank, there are M eigenvalues $[\lambda_1, \lambda_2, \dots, \lambda_M]$ corresponding to M eigenvectors $[\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_M]$. As $\boldsymbol{R}$ is Hermite matrix, all eigenvectors are orthogonal:

$$\boldsymbol{v}_i{}^H\boldsymbol{v}_j = 0 \quad i \neq j$$

Here there are M eigenvalues corresponding to the eigenvalues of matrix $\boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H$ $[\lambda_1{}', \lambda_2{}', \dots, \lambda_N{}']$ plus σ^2, and the rest $N - M$ eigenvectors are all σ^2.

$$[\lambda_1, \lambda_2, \dots, \lambda_M]\boldsymbol{I} = \begin{bmatrix} \lambda_1{}' + \sigma^2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_I{}' + \sigma^2 & 0 & \dots & 0 \\ 0 & \dots & 0 & \sigma^2 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \sigma^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{E_s} & 0 \\ 0 & \boldsymbol{E_n} \end{bmatrix}$$

It can be seen that σ^2 is the minimum value of eigenvalues in matrix $\boldsymbol{R}$. Sequencing the values of eigenvectors in descending order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M > 0$$

Here, former M bigger eigenvalues correspond to signal, latter $N - M$ smaller eigenvalues correspond to noise. Therefore, it is possible to separate the eigenvalues (eigenvectors) of matrix $\boldsymbol{R}$ to signal eigenvalues (eigenvectors in matrix $\boldsymbol{E_s}$) and noise eigenvalues (eigenvectors in matrix $\boldsymbol{E_n}$).

Assume λ_i is the *i-th* eigenvalue of matrix $\boldsymbol{R}$, and $\boldsymbol{v}_i$ is the eigenvector corresponding to λ_i:

$$\boldsymbol{R}\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$$

Let $\lambda_i = \sigma^2$

$$\boldsymbol{R}\boldsymbol{v}_i = \sigma^2\boldsymbol{v}_i \quad i = I + 1, I + 2, \dots, M$$

Recall

$$\boldsymbol{R} = \boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H + \sigma^2\boldsymbol{I}$$

So

$$(\boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H + \sigma^2\boldsymbol{I})\boldsymbol{v}_i = \sigma^2\boldsymbol{v}_i$$

Simplify

$$\boldsymbol{c}\boldsymbol{R}_s\boldsymbol{S}^H\boldsymbol{v}_i = \boldsymbol{0}$$

$$\left[\boldsymbol{R}_s^{-1}(\boldsymbol{S}^H\boldsymbol{S})^{-1}\boldsymbol{S}^H\right]\boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H\boldsymbol{v}_i = \left[\boldsymbol{R}_s^{-1}(\boldsymbol{S}^H\boldsymbol{S})^{-1}\boldsymbol{S}^H\right] \times \boldsymbol{0} = \boldsymbol{0}$$

$$\boldsymbol{S}^H\boldsymbol{v}_i = \boldsymbol{0} \qquad i = I+1, I+2, \dots, M$$

It can be shown that noise eigenvector $\boldsymbol{v}_i$ is orthogonal to the column vectors of matrix $\boldsymbol{S}$. Each column vectors of matrix $\boldsymbol{S}$ corresponds to the direction of signal. It shows the idea that to get the DOA of the signal from noise eigenvectors.

In terms of noise eigenvector matrix $\boldsymbol{E_n}$:

$$\boldsymbol{E_n} = [\boldsymbol{v}_I, \boldsymbol{v}_{I+1}, \dots, \boldsymbol{v}_M]$$

Here define MUSIC spatial spectrum:

$$P_{MUSIC}(\phi) = \frac{1}{\boldsymbol{s}^H(\phi)\boldsymbol{Q_n}\boldsymbol{Q_n}^H\boldsymbol{s}(\phi)} = \frac{1}{\left\|\boldsymbol{E_n}^H\boldsymbol{s}(\phi)\right\|^2}$$

In the expression, the denominator is the square of inner product of noise matrix $\boldsymbol{E_n}$ and signal vectors $\boldsymbol{s}(\phi)$. Ideally, when $\boldsymbol{E_n}$ and $\boldsymbol{s}(\phi)$ are orthogonal, denominator equal zero. Practically, due to the existence of noise, denominator reaches minimum value but not zero. So under this condition, $P_{MUSIC}(\phi)$ reaches peak value. Therefore, by searching different value of ϕ, the DOA of signals could be found by searching where peak values appear in the spectrum.

In summary, the steps to implement MUSIC algorithm are:

(1) Base on the number of samples of received signals to estimate the covariance matrix

$$\boldsymbol{R} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_k\,\mathbf{x}_k{}^H$$

Do eigendecomposition of covariance matrix $\boldsymbol{R}$:

$$\boldsymbol{R} = \boldsymbol{S}\boldsymbol{R}_s\boldsymbol{S}^H + \sigma^2\boldsymbol{I}$$

(2) Sequencing the values of eigenvectors in descending order, take the *M* bigger eigenvalues and eigenvectors as signal subspace $\boldsymbol{E_s}$ and take the $N-M$ smaller eigenvalues and eigenvectors as noise subspace $\boldsymbol{E_n}$:

$$\boldsymbol{S}^H\boldsymbol{v}_i = \boldsymbol{0} \qquad i = I+1, I+2, \dots, M$$

$$\boldsymbol{E_n} = [\boldsymbol{v}_I, \boldsymbol{v}_{I+1}, \dots, \boldsymbol{v}_M]$$

(3) Construct MUSIC spatial spectrum function:

$$P_{MUSIC}(\phi) = \frac{1}{\boldsymbol{s}^H(\phi)\boldsymbol{E_n}\boldsymbol{E_n}^H\boldsymbol{s}(\phi)} = \frac{1}{\left\|\boldsymbol{E_n}^H\boldsymbol{s}(\phi)\right\|^2}$$

Calculate the spectrum function $P_{MUSIC}(\phi)$ and search peak values in the spectrum which corresponding to DOA estimation of the signals.

Though MUSIC algorithm has many advantages, there are still shortcomings of this method. One problem is that if using estimated correlation matrix, the noise eigenvectors are no longer the same as the exact ones. Under this circumstance, the noise matrix is no longer strictly orthogonal to the signal matrix and leads to deviation, which has already mentioned above.

A significant problem by using MUSIC algorithm is the assumption that the signals are uncorrelated to each other. In practice, signals could be correlated or sufficiently close, thus the performance of MUSIC algorithm would deteriorate or even become invalid. Under such condition, some improvement methods must be made, such as reconstructing a conjugate matrix of matrix $\boldsymbol{X}$ and construct a noise subspace.

Another problem is the assumption made that the number of signal sources M must less than the number of array antennas N. This is necessary because MUSIC algorithm depends the noise subspace to estimate DOA of signals. Therefore there is always the restriction that $M < N$ when using MUSIC algorithm.

In terms of accuracy, MUSIC proved accurate DOA estimation. However, these are due to the fact that the number of snapshots is sufficient. The performance could also deteriorate when the signal observation period is limited.

Chapter 3

Proposed algorithm

Up to now, there are several methods that have been introduced in **Literature Survey**. It can be seen that different method has its advantages as well as disadvantages.

The most commonly concerned method should be conventional FFT-based method, due to its simple logic and high level of maturity.

Conventional FFT-based method would provide an accurate DOA estimation result under the condition that there is only one single signal source. For single source signal, even there is only one snapshot, the DOA estimator would still get an unbiased estimated result. However, to get such unbiased result, a very large value of zero-tapping multiple is required. Otherwise, there is no possibility to get the unbiased value.

In terms of the computational complexity of FFT operation, it could be shown that for an L points FFT, the total number of FFT operations is:

$$C_{FFT} = L \times \log_2 L$$

Where L is determined by the multiplication of number of array antennas M and zero-tapping multiple z:

$$L = z \times M$$

Therefore, it can be seen that with the increasing of the zero-tapping multiple z in order to get a sufficiently accurate result, the computational times is also increasing significantly, which would cost large CPU memories and processing time.

Therefore, to avoid this disadvantage under such condition, a more computational efficient algorithm could be valuable. The new algorithm should have at least the same accuracy as conventional method but cost less computational times. Higher accuracy is more desirable.

Besides, in terms of conventional FFT method, if there are more than two source signals, it will suffer the problem of hard or even fail to resolve the DOA of each source signal. This is because the significant sidelobe leakage in the spectrum. To abbreviate this problem, adding windowing function could be a solution, but this will lead a higher computational cost.

Therefore, a more efficient algorithm that could significantly reduce or remove the bias caused by spectral leakage but does not require higher computational cost is required.

In this section, a proposed algorithm[16] that suits for both single and multiple source signals which could effectively abbreviate the disadvantages above is introduced.

3.1 Single source Interpolator

For a single source signal, its spatial frequency ν could be expressed as following:

$$\nu = \frac{m + \sigma}{M}$$

Here, m is an integer number which ranges from

$$m \in [-\frac{M}{2}, \frac{M}{2} - 1]$$

And σ is the residual number which ranges from

$$\sigma \in [-0.5, +0.5]$$

Once the estimated spatial frequency value $\hat{\nu}$ is got, the estimated value of DOA could be got by equation:

$$\hat{\phi} = sin^{-1}(\frac{\lambda}{d}\hat{\nu})$$

To get the estimated value of spatial frequency $\hat{\nu}$, there are two stages that needs to be done:

(1) A first coarse stage is to do an M-point FFT and get its spatial spectrum without any zero-padding. This means the value of zero-tapping multiple *z* equals 1.

Once searching the M-point FFT spectrum, the coarse estimated value could be get from finding the integer index $\hat{\mu}$ which corresponds to peak value of the spectrum:

$$\hat{\mu} = \hat{m}, \qquad \hat{m} \in [-\frac{M}{2}, \frac{M}{2} - 1]$$

(2) The second step is to do an iterative calculation that to have a fine search, at the two points which one is positive 0.5 larger than integer index $\hat{\mu}$ and the other one is negative 0.5 smaller than integer index $\hat{\mu}$. Then doing the single point FFT at these two points and using these two frequency values to update the value of estimated value $\hat{\mu}$ iteratively. The equations of the iteration step could be expressed as followed:

- Calculate two points FFT value:

$$X[\hat{\mu} + r] = \sum_{k=0}^{M} x[k]e^{-j2\pi k\frac{\hat{\mu}+r}{M}}, \quad r = \pm 0.5$$

- Calculate the coefficient *h*:

$$h = \frac{1}{2}Re[\frac{X[\hat{\mu} + 0.5] + X[\hat{\mu} - 0.5]}{X[\hat{\mu} + 0.5] - X[\hat{\mu} - 0.5]}]$$

- Update the value of $\hat{\mu}$

$$\hat{\mu} = \hat{\mu} + \frac{\sin(\pi/M)}{\pi/M}h$$

By using this proposed algorithm, when doing the iterations for a sufficient times, the final estimated DOA of the source signal could provide a reasonable result.

3.2 Fast Iterative Estimator for multiple sources

As mentioned above, there should be an efficient algorithm that could significantly reduce or remove the bias caused by spectral leakage under multiple sources signals condition.

Here, similarly, an iterative algorithm is used to subtract and remove the spectral leakage terms produced in previous iteration.

For multiple source case, single snapshot circumstance, the total source number is *I*. Here

there are also two main steps of this algorithm:
(1) The first step is also to do a coarse M-point FFT and get its spectrum without zero-tapping (zero-tapping multiple *z* equals 1).
During this stage, for each source signal, just simply assume the estimated values of amplitude $\hat{\alpha}_i$ and frequency $\hat{\mu}_i$ all equal 0:

$$\hat{\alpha}_i = 0, \quad i = 0,1 \cdots, I$$
$$\hat{\mu}_i = 0, \quad i = 0,1 \cdots, I$$

(2) The second step is to do the iterative calculation which is also similar as single source case:
At the begging of the iteration loop, there is a initialization operation which is only done once:

$$\hat{X}_l[n] = X_l[n] - \sum_{i=1,i\neq l}^{I} \hat{\alpha}_i \hat{S}_i[n], \quad n = 0,1 \cdots, M-1$$

$$\hat{\mu}_l = \frac{1}{r} arg \ \max_{n} \left|\hat{X}_l[n]\right|^2$$

Here $l = 1,2 \cdots, I$, which means to calculate each source signal one by one.
After finishing the initialization, the following step should be calculated iteratively.

$$\hat{X}_l[r] = X_l[r] - \sum_{i=1,i\neq l}^{I} \hat{\alpha}_i \hat{S}_i[\hat{\mu}_l + r], \quad r = \pm 0.5$$

Here $\hat{S}_i[\hat{\mu}_l + r]$ is the leakage DFT term and it can be calculated as:

$$\hat{S}_i[\hat{\mu}_l + r] = \sum_{k=0}^{M-1} s_i[k]\, e^{-j2\pi k \frac{\hat{\mu}_l + r}{M}} = \frac{1 + e^{j2\pi(\hat{\mu}_i - \hat{\mu}_l)}}{1 + e^{j2\pi(\hat{\mu}_i - \hat{\mu}_l + r)}}$$

Using the calculated two points value above:

$$h_l = \frac{1}{2} Re[\frac{X[0.5] + X[-0.5]}{X[0.5] - X[-0.5]}]$$

Update $\hat{\mu}_l$:

$$\hat{\mu}_l = \hat{\mu}_l + \frac{\sin(\pi/M)}{\pi/M} h_l$$

Finally update $\hat{\alpha}_l$

$$\hat{\alpha}_l = \frac{1}{M}\left\{\sum_{k=0}^{M} x[k] e^{-j2\pi\frac{\hat{\mu}_l}{M}} - \sum_{i=1,i\neq l}^{I} \hat{\alpha}_i \hat{S}_i[\hat{\mu}_l]\right\}$$

The whole procedure will stop after the specified number of iterations.

3.3 Computational complexity

As mentioned above, the proposed algorithm should provide a more efficient way to do the iterative calculation. The following is the theoretical comparison in terms of the computational times required between proposed algorithm and conventional method.

- In terms of conventional FFT-based method, to get a reasonable accurate result, it is required to a zero-padding and the total computational cost of *L-points* FFT multiplication is:

$$C_{FFT} = L \times \log_2 L = zM \times \log_2(zM)$$

- In terms of the proposed algorithm, because in the first step, there is no requirement to do the zero-padding before doing FFT, therefore the computational cost for the *M-points* coarse step calculation is:

  $$C_{proposed\_1} = M \times \log_2 M$$

  Additionally, during each iteration loop, for each single source , it is required to do an additional two points' FFT calculation. For each spatial frequency point, doing *M-points* FFT calculation will require M times FFT multiplication. Therefore, to do each iteration calculation, it requires total additional computational time which is:

  $$C_{proposed\_2} = 2 \times M$$

  Overall, the required FFT multiplication times of proposed algorithm is:

  $$C_{proposed} = C_{proposed_1} + C_{proposed_2} = M \times (2 + \log_2 M) = M \times \log_2(4M)$$

Compared the result with the computational times required for conventional method:

$$C_{FFT} = M \times z \log_2(zM)$$

As long as the zero-padding multiple value $z \geq 4$, the proposed algorithm will always provide less computational cost compared to conventional method. And generally, the number of antennas M in the automotive is limited, to get a reasonable high accuracy by using conventional method, sufficiently large value of zero-padding multiple is needed, which is significantly larger than 4. Therefore, the proposed algorithm would has a significantly lower computational cost. This advantage becomes more obvious with the increasing of the zero-padding multiple z.

Chapter 4

Implementation Result

4.1 Capon Beamforming

Capon Beamforming implementation is the work I did in Thesis A. The advantage of this method is, it has a simple logic and easy to be implemented in simulation. It is a good start to investigate this method and have an initial understanding of what DOA estimation is doing and how the angle value could be obtained by searching the peak value of spatial spectrum.

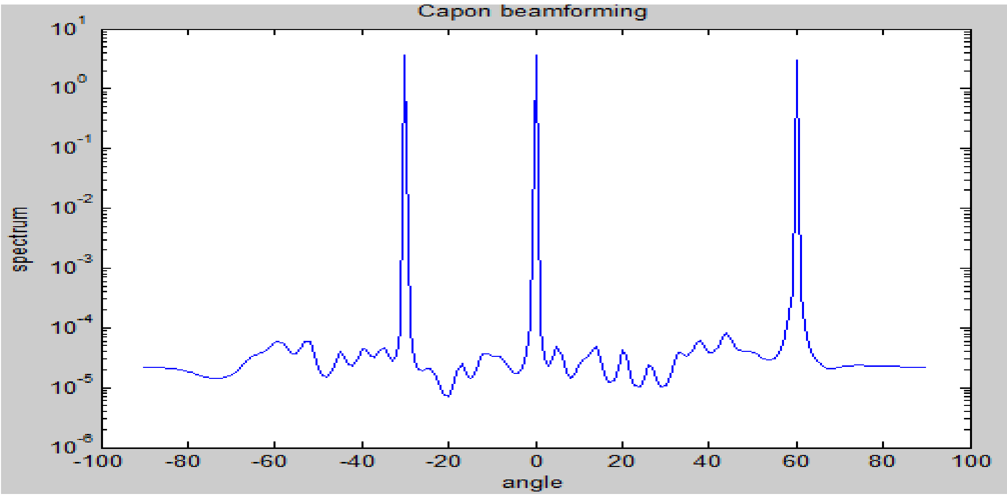

Figure 4.1.1 $I = 3, M = 32, SNR = 20, \phi_1 = -30°, \phi_2 = 0°, \phi_3 = 60°$

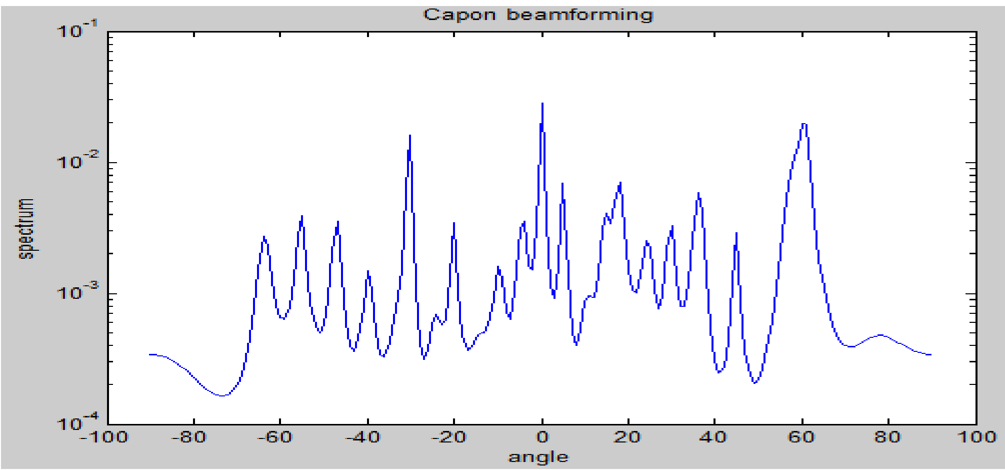

Figure 4.1.2 $I = 3, M = 32, SNR = 0, \phi_1 = -30°, \phi_2 = 0°, \phi_3 = 60°$

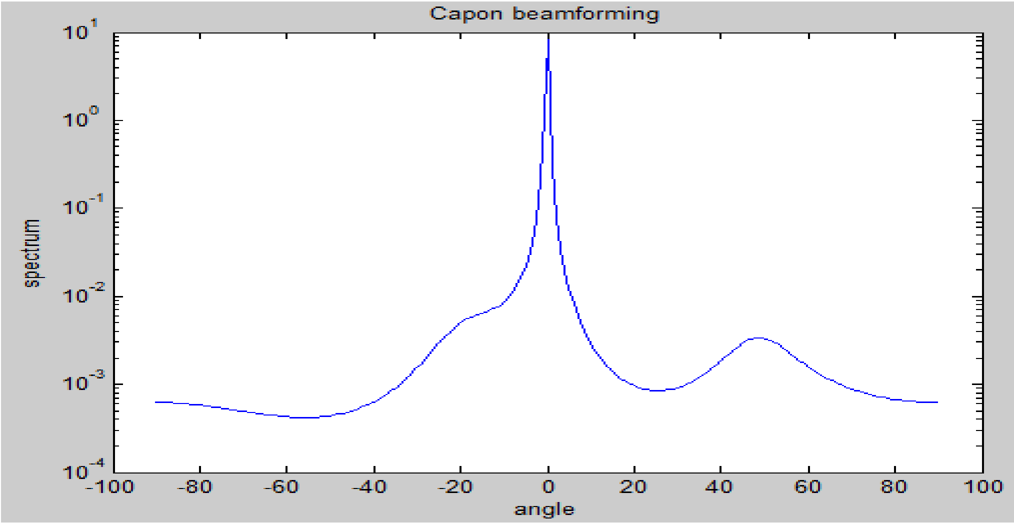

Figure 4.1.3 $I = 3, M = 4, SNR = 20, \phi_1 = -30°, \phi_2 = 0°, \phi_3 = 60°$

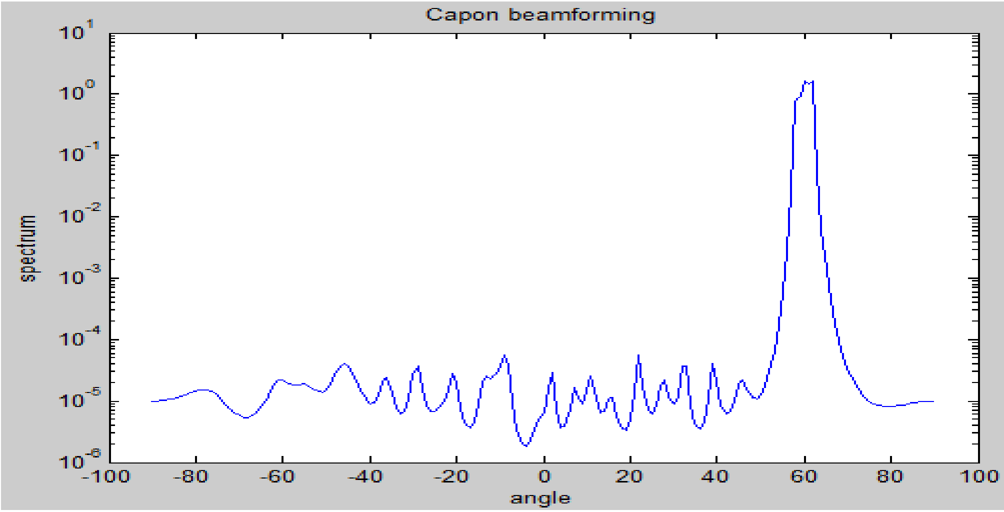

Figure 4.1.4 $I = 3, M = 32, SNR = 20, \phi_1 = 58°, \phi_2 = 60°, \phi_3 = 62°$

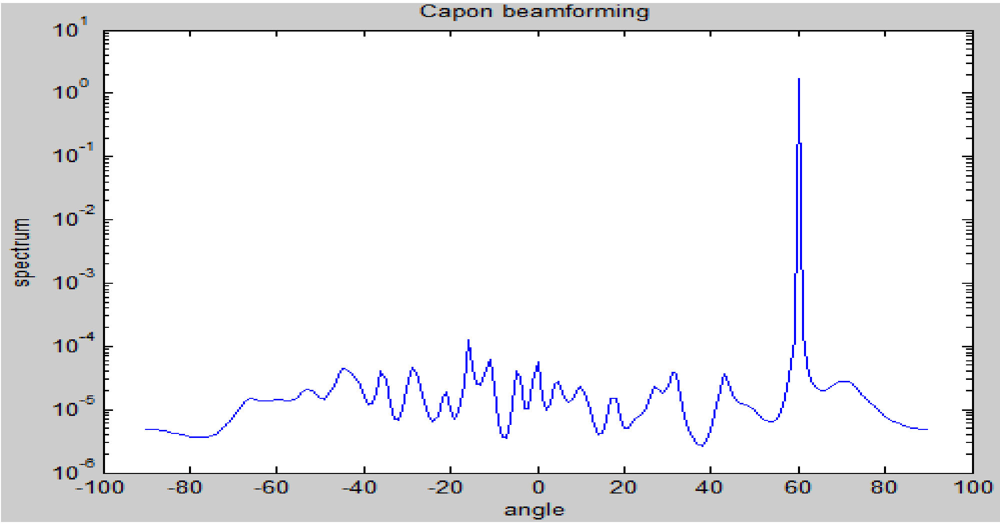

Figure 4.1.5 $I = 1, M = 32, SNR = 20, \phi_1 = 60°$

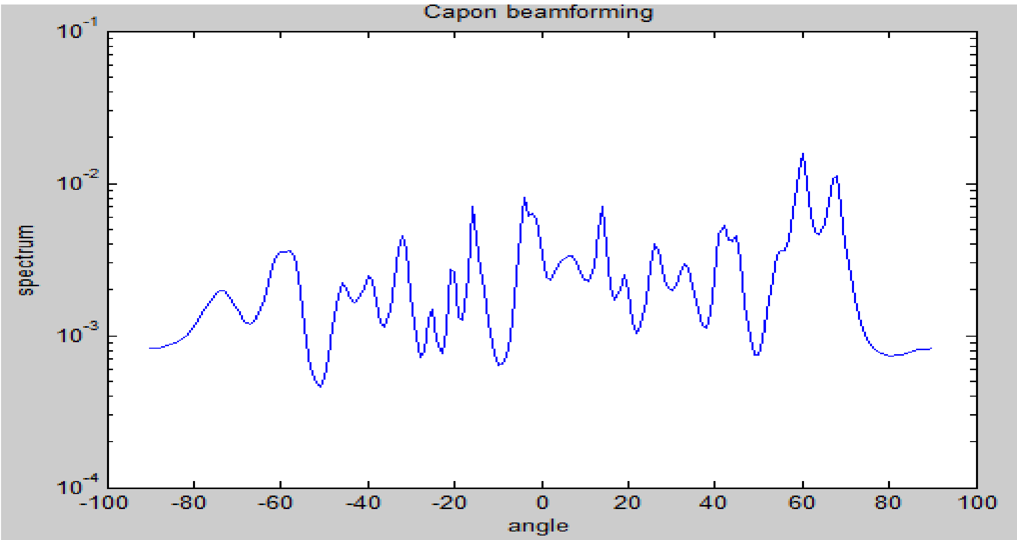

Figure 4.1.6 $\boldsymbol{I = 1, M = 32, SNR = 0, \phi_1 = 60°}$

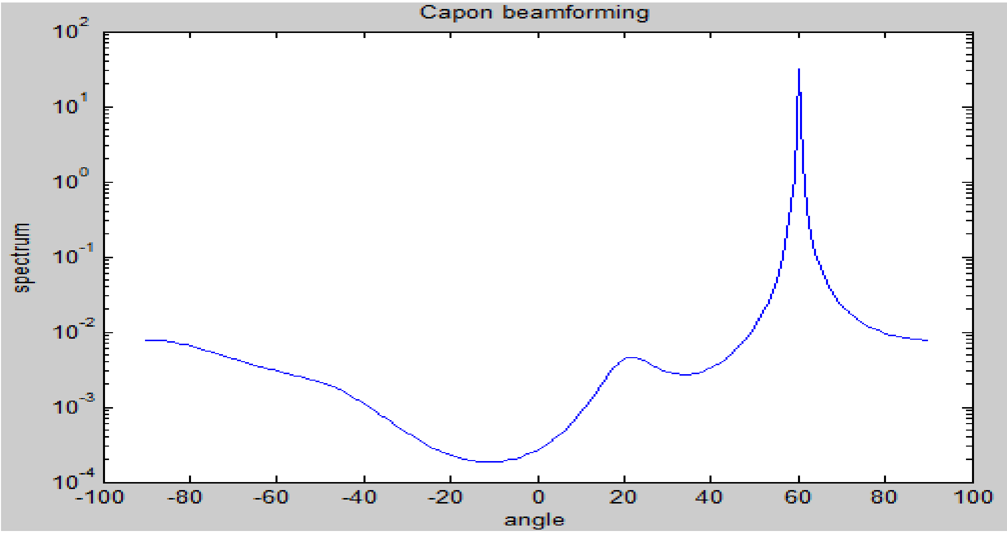

Figure 4.1.7 $\boldsymbol{I = 1, M = 4, SNR = 20, \phi_1 = 60°}$

During simulation, I tested different conditions by modifying the parameters of:

(1) *SNR* value

(2) Number of array antennas M

(3) Direction angle of source signals ϕ

(4) Number of source signals I

Figure 4.1.1 shows the condition that there are $I = 3$ source signals, with direction angles $\phi_1 = -30°, \phi_2 = 0°, \phi_3 = 60°$, respectively. SNR for each signal is *20*. The number of array antennas $M = 32$.

It can be seen that under such condition, the DOA of the desired three signals could be clearly identified by searching the three peak values from spectrum. And the angle of the three peak values match the actual directions.

Figure 4.1.2 shows the condition that reducing the SNR from 20 to 0, all the other parameters remain unchanged. It can be seen under such condition, spectrum performance deteriorate and it becomes harder to identify the DOA of signal in the spectrum because of the noise interference.

Figure 4.1.3 shows the condition that reducing the antenna number from $M = 32$ to $M = 4$. All the other parameters remain the same as **Figure 4.1.1**. Under such condition, it can be seen that it fails to resolve the signals and only one peak value appears at 0.

Figure 4.1.4 shows the condition that setting three signals as close targets. The direction angles of them are $\phi_1 = 58°, \phi_2 = 60°, \phi_3 = 62°$. All the other parameters remain the same as **Figure 4.1.1**. Under such condition, it can be seen that there is only one peak appears at around $60°$. Theoretically, there should be three close peak values appeared. Practically, it fails to resolve the targets because they are too close.

Figure 4.1.5 is the single target situation. I set signal number $I = 1$ and direction angle $\phi_1 = 60°$. Other parameters remain the same as **Figure 4.1.1**. Under such condition, it is clearly to detect the DOA of the target signal in the spectrum.

Figure 4.1.6 shows the condition that reducing SNR from 20 to 0. Other parameters remain the same as **Figure 4.1.5**. Under such condition, it can be seen that the DOA of the signal could not be clearly identified in the spectrum because of significant noise.

Figure 4.1.7 shows the condition that reducing the number of antennas from $M = 32$ to $M = 4$. Other parameters remain the same as **Figure 4.1.5**. Under such condition, because there is only one signal source, it is still capable to detect the correct DOA of that signal.

In conclusion of Capon beamforming method:

(1) SNR is an important factor to determine the result. The higher the SNR, the easier to detect the DOA of the signal.
(2) Number of antennas is another important factor. The more antennas, the better result.
(3) Whether source signals are close or not could be a factor need to be considered. Too close targets may not be resolved by this method.
(4) Only one single target is more easier to be detected by comparison with multiple targets situation.

Actually, what I got from my conclusion matches the conclusion in the **Literature Survey, 2.5 Capon Beamforming**.

4.2 Conventional FFT-based method

In this section, I tried to implement the conventional FFT-basad method.

The reason I implement this method is:

1. Conventional FFT-based method is a quite typical approach in DOA estimation, it has specific advantages as well as disadvantages. It provides the basic concept of how DOA estimation works. Additionally, conventional FFT method is relatively easier to understand and coding in program. By implementing this method, I could have a better understanding of DOA estimation.

2. Finishing implement conventional FFT-based method could give me a result which is useful when I implement the proposed algorithm later on. The result could be seen as reference when I compare the performance between conventional FFT and proposed method.

Single source estimation

The following figures show the simulation result for single source case:

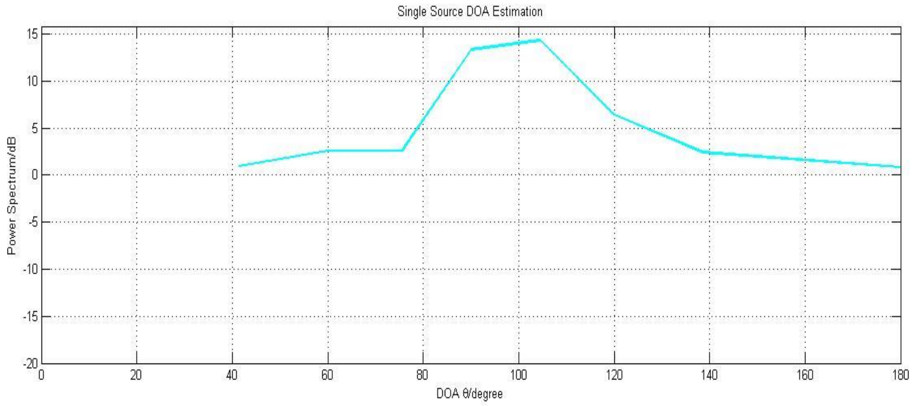

Figure 4.2.1 Single source conventional FFT method with z=1

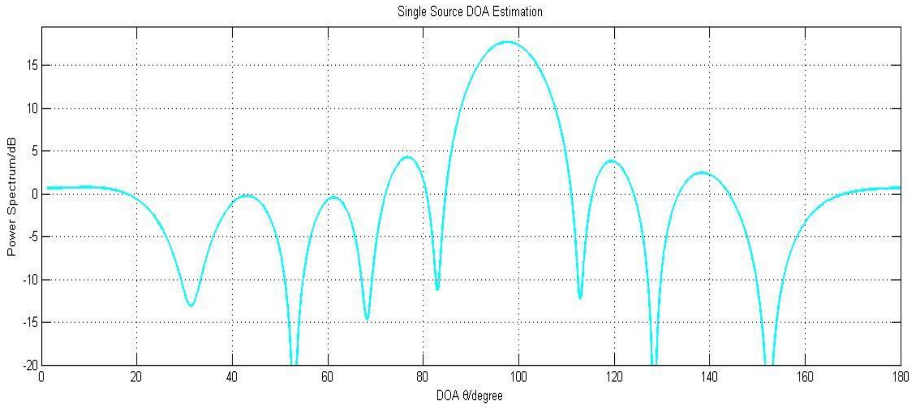

Figure 4.2.2 Single source conventional FFT method with z=1024

Figure 4.2.1 and **Figure 4.2.2** shows the comparison of different values of zero-tapping multiple used, under the single source condition.

It can be seen from **Figure 4.2.1**, there is only $M = 8$ points FFT in the spectrum. So the resolution of the spectrum could no more than $2\pi/M$ under this condition. Once found the peak value in the spectrum, the corresponded spatial frequency is the estimated frequency value. Without any zero-tapping, the result could be biased and the maximum deviation could be as high as π/M, which is very significant when M is small. Therefore, sufficient zero-tapping is needed to reduce the bias. **Figure 4.2.2** has the zero-tapping multiple *z=1024*, which provide a significant better estimation result.

By comparing the calculate result with the pre-set true DOA value:

For **Figure 4.2.1**, the final result caused a deviation which is 11.3096°

For **Figure 4.2.2,** the final result caused a deviation which is 0.1019°

The calculation is based on $M = 8$ antenna arrays and the signal to noise ratio $SNR = 20\ dB$.

It is worth pointed out that with the increasing of *SNR*, the deviation would drop accordingly.

But it is not practical for the real case, because the *SNR* is determined by the environment and could not been changed by human being.

Multiple sources estimation

The following figures show the simulation results for multiple sources case:

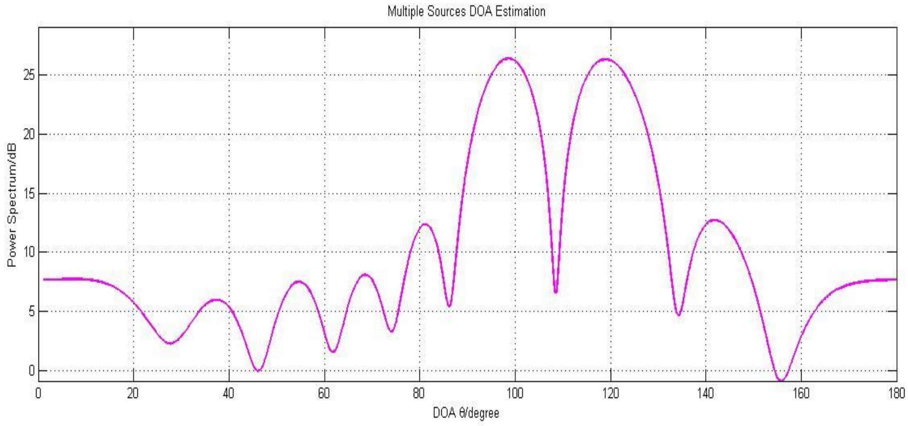

Figure 4.2.3 Two sources conventional FFT method with sufficiently far direction angle

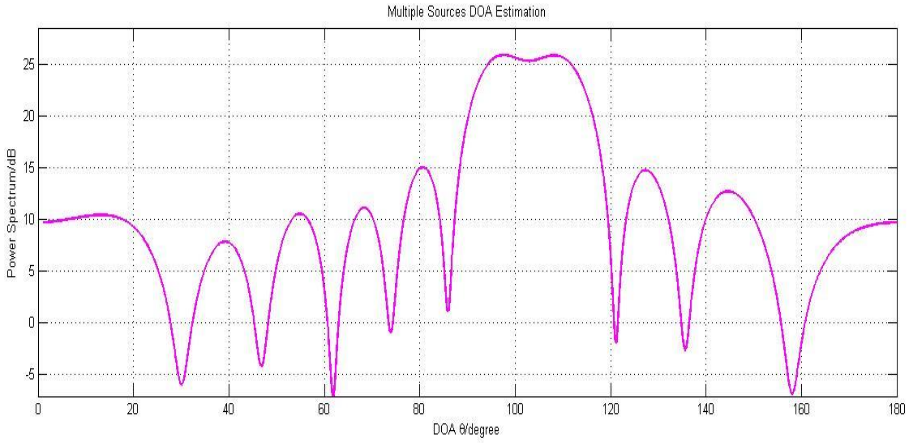

Figure 4.2.4 Two sources conventional FFT method with close direction angle

Figure 4.2.3 and **Figure 4.2.4** shows the performance by comparison of different direction between two signal sources. The zero-tapping multiple used here is always *z=1024*. The number of antennas is $M = 10$ antenna arrays and the signal to noise ratio $SNR = 20\ dB$ for all source signals.

In **Figure 4.2.3**, the difference of direction angle between the two source signals is set to 30°, which is larger than $\pi/M = 18°$. Under such situation, the estimator still has the ability to resolve the two targets. It could be clearly seen the two peak values in the spectrum, which represents the two signal sources' spatial frequency, respectively.

In **Figure 4.2.4**, the difference of direction angle between the two source signals is set to 16.5°, which is less than $\pi/M = 18°$. Under such situation, the estimator almost fails to resolve the two targets. It could be seen from the spectrum that the two peak values are merged together and it is hard to find the two peak values already. With further reducing the direction difference, the final spectrum would fall into only one peak, which means totally fail to estimate the correct DOA of source signals.

Brief Conclusion

From the conventional FFT-based method implementation, it can be shown that this method is more effective for the single source case, which would provide an unbiased estimation, but at the cost of high computational cost due to large multiple of zero-padding. For multiple source case, it suffers the problem of significant spectral leakage which could cause biased estimation. These conclusions match the conclusions mentioned in **Literature Survey, 2.6 Conventional FFT-based DOA Estimation**.

4.3 MUSIC algorithm

MUSIC algorithm is also a quite typical and famous algorithm in the field of subspace method. It is worth to investigate the details of this algorithm to have a better understanding of the differences between subspace methods compared to conventional DOA methods.
The following figure gives a spectrum analysis of MUSIC algorithm

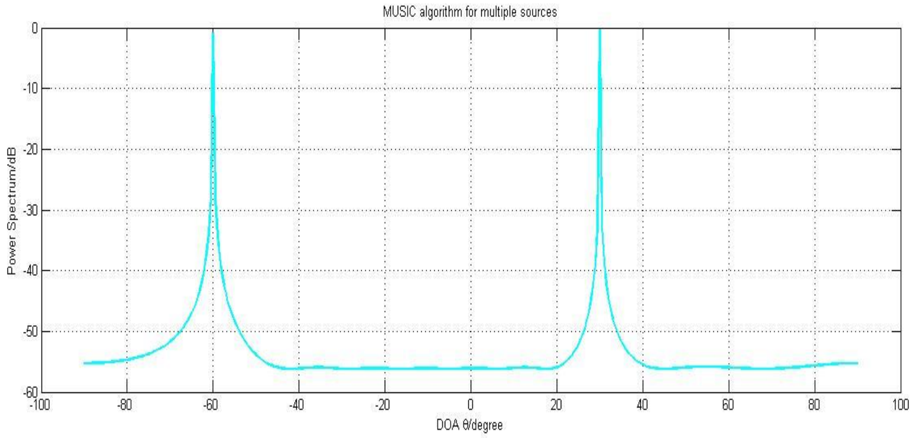

Figure 4.3.1 Two source signals MUSIC algorithm

Figure 4.3.1 shows the situation for two source signals DOA estimation by using MUSIC algorithm. From the spectrum, there are two distinctive peak values, which correspond to two sources DOA, respectively. It could be shown that MUSIC algorithm does not have the problem of spectral leakage and provides an accurate estimated result.
The following work is to further analyse how different parameters could affect the performance of MUSIC algorithm, finding the characteristics as well as the limitation. If not specified, the default value of: Antennas number $M = 10$, Signal to Noise Ratio $SNR = 20\ dB$, Number of snapshots $N = 200$.

Different Number of snapshots *N*

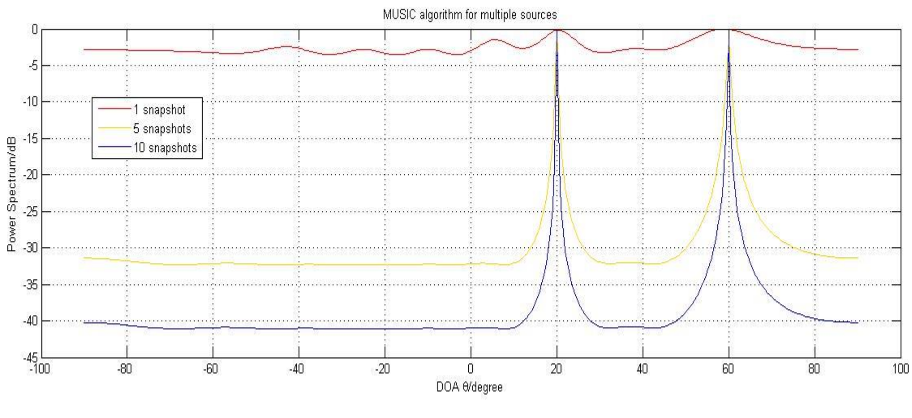

Figure 4.3.2 Relationship of Number of snapshots N in MUSIC

Figure 4.3.2 shows the relationship between spectrum performance and different number of snapshot N. For the red line, which is the situation that there is only one single snapshots of the received signal, it can be seen that the estimated spectrum is quite flat and therefore hard to detect the correct peak value from spectrum. With the increasing of the number of snapshot N, the estimated result is becoming more reliable, accordingly. This proves the fact that MUSIC algorithm is largely depending on the sufficient number of snapshot to get a good estimation result.

Different Number of antennas M

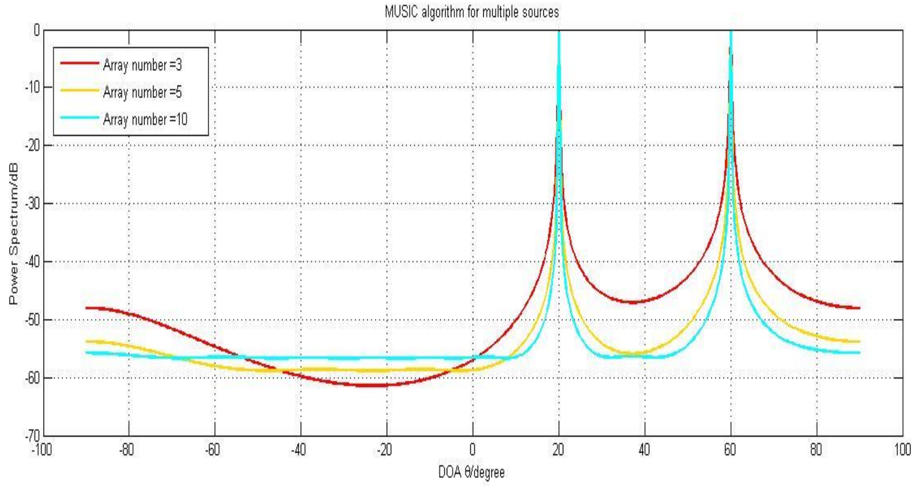

Figure 4.3.3 Relationship of Number of antennas M in MUSIC

Figure 4.3.3 shows the relationship between spectrum performance and different number of antennas M. It should be pointed out that during the simulation, there are three signals sources used in my model. So based on the theory that the number of antennas M must be no less than the number of signal sources I in order to get noise subspace vectors. Therefore, the least tested array antenna numbers is $M = 3$. It could be seen even under the condition that there are only minimum number of antennas, the MUSIC algorithm still could get an accurate estimation of desired signals' DOA.

Different value of SNR

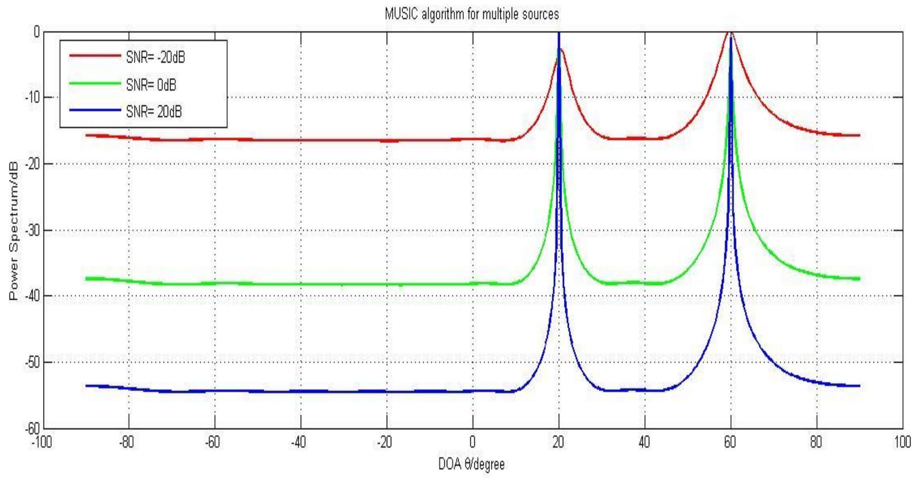

Figure 4.3.4 Relationship of value of SNR in MUSIC

Figure 4.3.4 shows the relationship between spectrum performance and different value of SNR. It can be seen that even under the condition $SNR = -20\ dB$ (which means the energy of desired source signal is only 1% of the noise energy), MUSIC algorithm still has the ability to detect the peak value in the spatial spectrum, which means it has a strong advantage when working in noisy environment. However, from the red line, it could be seen a small bias on the left peak of the line. This means the estimated value is actually biased. With the increasing of the SNR, the estimated value become accurate and unbiased.

Different value of direction angle difference

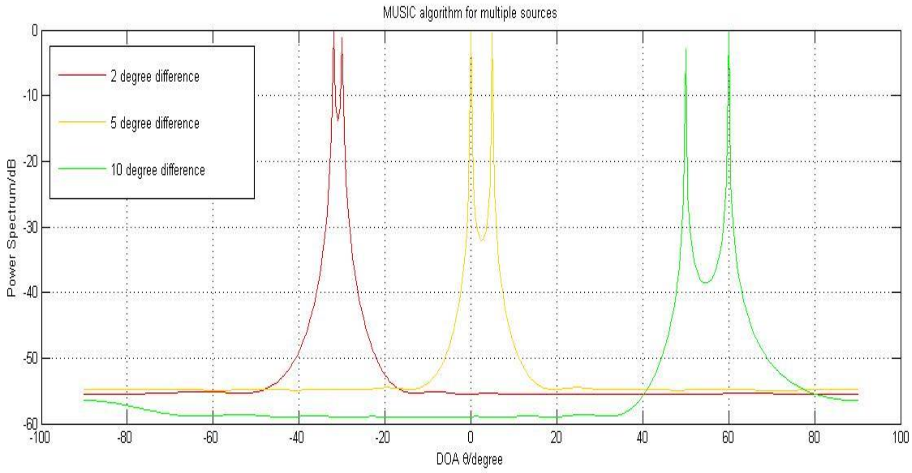

Figure 4.3.5 Relationship of direction angle difference in MUSIC

Figure 4.3.5 shows the relationship between spectrum performance and the difference between two signal direction angle. It can be seen under the situation that the $\pi/M = 18°$, even when the angle difference is only 2°, MUSIC algorithm could still successfully resolve these two close targets.

Brief Conclusion

From all the results above, it could be seen that MUSIC is a very accurate algorithm which does not suffer the problem of spectral leakage. In comparison with conventional FFT-based method, MUSIC provides good estimation result even under the condition that the direction difference between two targets is less than π/M.

However, the MUSIC algorithm will not work well when the snapshot number is reduced to a small value. The dependence on large number of snapshot would cause a increasing of computational complexity.

4.4 Proposed algorithm

In this section, I implement the proposed algorithm as mentioned in **Chapter 3**. To compare the improvement of proposed algorithm with conventional FFT method, there are three different aspects I specifically looked at:

1. Relationship between *SNR* value and final estimated accuracy.
2. Relationship between number of antennas *M* and final estimated accuracy.
3. Relationship between number of antennas *M* and required computation times.

If not specified, the default values used in this simulation are:
Antennas number $M = 10$.
Signal to Noise Ratio $SNR = 20\ dB$.
Zero-tapping multiple of conventional FFT method $z = 4$

Relationship between *SNR* value and final estimated accuracy
The simulation runs 5000 times independently.

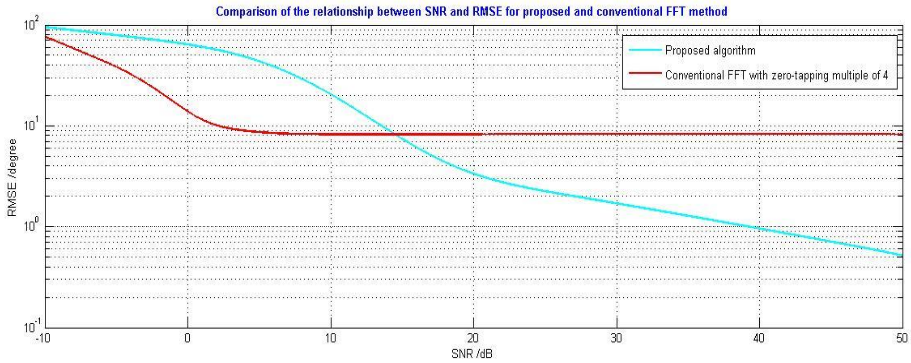

Figure 4.4.1 *SNR* vs. *RMSE* for proposed algorithm and conventional FFT

Figure 4.4.1 shows the comparison of different value of *SNR* versus final *RMSE* in degree. From the figure, it can be shown that when the *SNR* is low, usually below *15dB*, the *RMSE* of conventional FFT method is slightly less than the *RMSE* of proposed algorithm. With the increasing of the *SNR*, the *RMSE* of proposed algorithm becomes less than the *RMSE* of conventional FFT method. This difference becomes significant when the *SNR* becomes reasonably high.

It needs to be pointed out that the actual plot I got is not the same as **Figure 4.4.1**. There contains a lot of ripples in the original plot. I used a smooth filter to process the data and then print the figure out. In the original plot, the noise dominates the ripple in low *SNR*, therefore, the value of *RMSE* in low *SNR* is not reliable. When using smooth filter, there would be some negative effect that change the origin relationship between proposed algorithm and conventional FFT method in low *SNR* band.

Relationship between number of antennas *M* and final estimated accuracy
The simulation runs 2000 times independently.

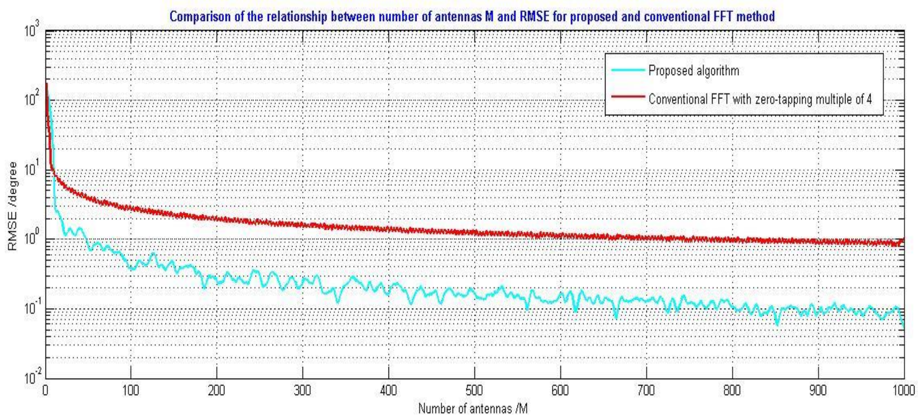

Figure 4.4.2 Antenna number *M* vs. *RMSE* for proposed algorithm and conventional FFT

Figure 4.4.2 shows the comparison of different number of antennas M versus final $RMSE$ in degree. The antenna number ranges from 1 to 1000. From the figure, it can be seen that for the same number of antenna number, the final $RMSE$ of proposed algorithm is always lower than that of conventional FFT method. This indicates that the proposed solution has a better accuracy for estimating the source signal DOA.

Relationship between number of antennas M and required computation times

The simulation runs 5000 times independently for both $z = 4$ and $z = 1024$.

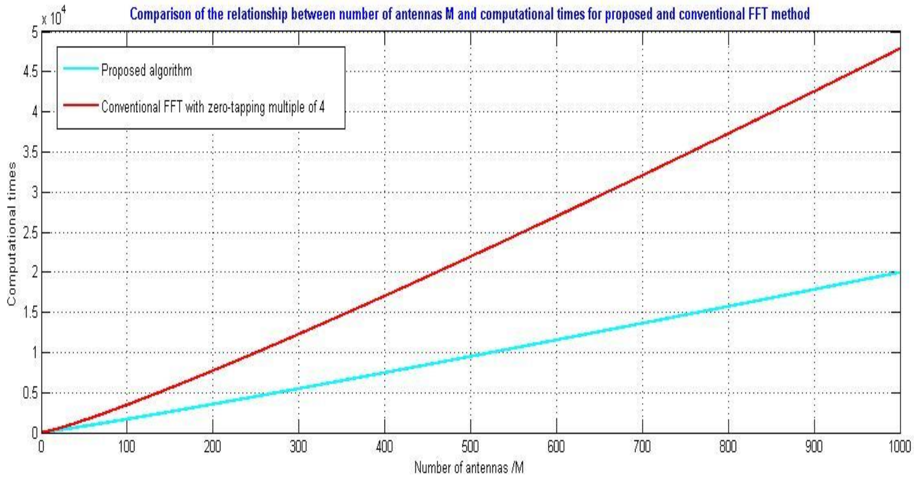

Figure 4.4.3 Antenna number M vs. Computational times when $z = 4$

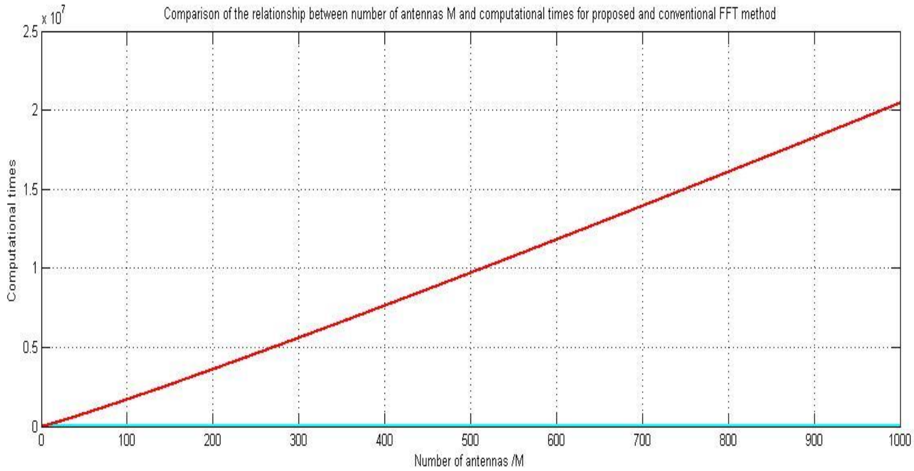

Figure 4.4.4 Antenna number M vs. Computational times when $z = 1024$

Figure 4.4.3 and **Figure 4.4.4** show the comparison of different number of antennas M versus required computational times under the conditions that $z = 4$ and $z = 1024$, respectively. From **Figure 4.4.3**, it could be seen that the computational times of proposed solution is less than the computational times of conventional FFT method. This difference increases as the number of antennas increases. **Figure 4.4.4** further shows how proposed algorithm significantly reduce computational cost with increasing the value of zero-padding multiple z.

Brief Conclusion

The proposed solution is based on the conventional FFT method, but by doing the coarse FFT at first step without zero-tapping and doing a fine and iterative searching at second step, the final accuracy is improved by comparison with conventional FFT method. Moreover, the computational times is significantly reduced by comparison with conventional FFT method, especially under the case that the zero-padding multiple z is sufficiently large.

Therefore, it is a successful improvement of DOA estimation conventional FFT method.

Chapter 5

Conclusions

This paper shows the most typically used methods in the field of DOA estimation. Among those methods, the most commonly concerned methods are conventional FFT-based DOA estimation and MUSIC algorithm. Some implementations of these two methods are done and the results are shown. Based on the conventional FFT-based method, a proposed algorithm is shown after, which aims to reduce the computational complexity under the same condition and provides better DOA estimation performance. Simulation results of the proposed solution are shown and the comparison is made between the proposed solution and conventional FFT method.

Appendix A- code for conventional-FFT

```
%% FFT based DOA estimation
%% single source, no window function
clc
close all
clear all

lamda=1;
d=lamda/2;

N=8; % array number
M=1; % number of sources
rad=pi/180;
theta_1=30.1; % DOA of source 1
theta_1=theta_1*rad;
phi_1=2*pi*(d*cos(theta_1)/lamda);
n=0:1:(N-1);
n=n';
s_1=exp(1i*n*phi_1);
SNR=20;
Ps=s_1'*s_1;
Pn=Ps*10^(-SNR/10);
sigma=sqrt(Pn/(2*N));
x=s_1+sigma*(randn(N,1)+1j*rand(N,1));

figure(1)
subplot(2,1,1)
plot(abs(s_1))
subplot(2,1,2)
plot(abs(x))

z=1024;
L=z*N;

X=fft(x,L);
Y=fftshift(abs(X).^2);
phi_axis=-180:360/L:180-360/L;
theta_axis=acosd(phi_axis/360*lamda/d);
[A,m]=max(Y);

figure(2)
```

```
plot(phi_axis,10*log10(fftshift(Y)),'LineWidth',2);
axis([-180 180 -20 1.1*max(10*log10(fftshift(Y)))])
xlabel('DOA \theta/degree')
ylabel('Power Spectrum/dB')
title('Single Source DOA Estimation')
grid on
phi_0=phi_axis(m)

figure(3)
plot(theta_axis,10*log10(fftshift(Y)),'LineWidth',2);
axis([0 180 -20 1.1*max(10*log10(fftshift(Y)))])
xlabel('DOA \theta/degree')
ylabel('Power Spectrum/dB')
title('Single Source DOA Estimation')
grid on
theta_0=theta_axis(m)

% X_axis=acos(x_axis*rad/(2*pi)*lamda/d)/rad
% figure(2)
% plot(spec)
% axis([0 360 min(spec) 1.2*max(spec)])
% grid on

%% Multiple targets
clc
close all
clear all

lamda=1;
d=lamda/2;

M=10; % array number
Source=[1 ; exp(1i*pi/4)]; % number of sources
rad=pi/180;

theta_1=30; % DOA of source 1
theta_1=theta_1*rad;
phi_1=2*pi*(d*cos(theta_1)/lamda);
theta_2=46.5; % DOA of source 2
theta_2=theta_2*rad;
phi_2=2*pi*(d*cos(theta_2)/lamda);

M_array=0:1:(M-1);
M_array=M_array';
```

```
s_1=exp(1i*M_array*phi_1);
s_2=exp(1i*M_array*phi_2);
s=[s_1 s_2];

ss=s*Source;

SNR=20;

x=ss+awgn(ss,SNR);

z=1024;
L=z*M;

X=fft(x,L);
Y=fftshift(abs(X).^2);
phi_axis=-180:360/L:180-360/L;
theta_axis=acosd(phi_axis/360*lamda/d);
[A,m]=max(Y);

figure(2)
plot(phi_axis,10*log10(fftshift(Y)));
axis([-180 180 min(10*log10(fftshift(Y)))
1.1*max(10*log10(fftshift(Y)))])
xlabel('DOA \theta/degree')
ylabel('Power Spectrum/dB')
title('Multiple Sources DOA Estimation')
grid on
phi_0=phi_axis(m)

figure(3)
plot(theta_axis,10*log10(fftshift(Y)));
axis([0 180 min(10*log10(fftshift(Y)))
1.1*max(10*log10(fftshift(Y)))])
xlabel('DOA \theta/degree')
ylabel('Power Spectrum/dB')
title('Multiple Sources DOA Estimation')
grid on
theta_0=theta_axis(m)
```

Appendix B- code for conventional-MUSIC

```
%% MUSIC Algorithm
%% General
clc
close all
clear all

N=200;%Snapshot
doa=[30 -60]/180*pi;%DOA
w=[pi/4 pi/3]';%frequency
M=10;%Arrary Numbers
P=length(w);
lambda=150;
d=lambda/2;%array element space
snr=20;
B=zeros(P,M);
for k=1:1:P
    B(k,:)=exp(-j*2*pi*d*sin(doa(k))/lambda*[0:M-1]);
end
B=B';
xx=2*exp(j*(w*[1:N]));
x=B*xx;
x=x+awgn(x,snr);%Gaussin noise
R=x*x';
[U,V]=eig(R);
UU=U(:,1:M-P);%noise sub space
theta=-90:0.5:90;
for ii=1:length(theta)
    AA=zeros(1,length(M));
    for jj=0:M-1
        AA(1+jj)=exp(-j*2*jj*pi*d*sin(theta(ii)/180*pi)/lambda);
    end
    WW=AA*UU*UU'*AA';
    Pmusic(ii)=abs(1/WW);
end
Pmusic=10*log10(Pmusic/max(Pmusic));%spatial spectrum
plot(theta,Pmusic,'-k','linewidth',2.0)
xlabel('DOA \theta/degree')
ylabel('Power Spectrum/dB')
title('MUSIC algorithm for multiple sources')
grid on
```

Appendix C- code for proposed algorithm

```
close all
clear all
clc

lamda=1;
d=lamda/2;

M=8; % array number
rad=pi/180;
theta_1=30.1; % DOA of source 1
theta_1=theta_1*rad;
phi_1=2*pi*(d*cos(theta_1)/lamda);
n=0:1:(M-1);
n=n';
s_1=exp(1i*n*phi_1);
SNR=20;
Ps=s_1'*s_1;
Pn=Ps*10^(-SNR/10);
sigma=sqrt(Pn/(2*M));
x=s_1+sigma*(randn(M,1)+1j*rand(M,1));

figure(1)
subplot(2,1,1)
plot(abs(s_1))
subplot(2,1,2)
plot(abs(x))

% z=1024;
% L=z*M;
L=M;

X=fft(x,L);
Y=fftshift(abs(X).^2);
phi_axis=-180:360/L:180-360/L;
theta_axis=acosd(phi_axis/360*lamda/d);
[A,m]=max(Y);

figure(2)
plot(phi_axis,10*log10(fftshift(Y)));
phi_0=phi_axis(m)
```

```
figure(3)
plot(theta_axis,10*log10(fftshift(Y)));
theta_0=theta_axis(m)

%%%%% iteration %%%%%%%
p=0.5;
h=0;
u=m-1/2*M-1;
X_p_pos=0;
X_p_neg=0;
Q=5;
for q=1:1:Q
  for k=0:1:M-1
      X_p_pos=X_p_pos+x(k+1)*exp(-1j*2*pi*k*(u+p)/M);
      X_p_neg=X_p_neg+x(k+1)*exp(-1j*2*pi*k*(u-p)/M);
  end
  h=1/2*real((X_p_pos+X_p_neg)/(X_p_pos-X_p_neg));
  u=u+(sin(pi/M))*h/(pi/M);
  X_p_pos=0;
  X_p_neg=0;
end
theta_xxx=acosd(u/M*lamda/d)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% X_axis=acos(x_axis*rad/(2*pi)*lamda/d)/rad
% figure(2)
% plot(spec)
% axis([0 360 min(spec) 1.2*max(spec)])
% grid on
```

Appendix D- code for Capon Beamforming

```
%% CB
clc
close all
clear all

i=sqrt(-1);
j=i;
degrad=pi/180;

N=4;
M=3;

f0=40;
f1=50;
f2=60;

nn=4;

phi_1=60;
phi_2=0;
phi_3=60;
phi=[phi_1]';

SNR=20;
SN1=SNR;
SN2=SNR;
SN3=SNR;
sn=[SN1];

tt=0:1/nn:1-1/nn;
x0=exp(-j*2*pi*f0*tt);
x1=exp(-j*2*pi*f1*tt);
x2=exp(-j*2*pi*f2*tt);
S=[x0];

Ps=S*S'./nn;
ps=diag(Ps);
refp=10.^(sn/10);
tmp=sqrt(refp./ps);
S2=diag(tmp)*S;
```

```
tmp=-j*pi*sin(phi*degrad);
tmp2=[0:N-1]';
a2=tmp2*tmp;
A=exp(a2);

nr=randn(N,nn);
ni=randn(N,nn);
u=nr+j*ni;

X=A*S2+(1/(10^(SNR/20)))*u;

Rxx=X*X'/nn;
invRxx=inv(Rxx);

theta=[-90:90]';
tmp=-j*pi*sin(theta'*degrad);
tmp2=[0:N-1]';
a2=tmp2*tmp;
A2=exp(a2);
den=diag(A2'*invRxx*A2);
doa=abs(1./den);

semilogy(theta,doa,'-blue');
title('Capon beamforming');
xlabel('DOA angle');
ylabel('spectrum');
grid on
```

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