Per-User Packet Outage Analysis in Slow Multiaccess Fading Channels with Successive Interference Cancellation for Equal Rate Applications

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***Abstract*— In this paper, we derive analytically the *per-user* packet outage probability and the total system goodput for multi-access systems using multiuser detector with adaptive successive interference cancellation (MUD-SIC). We consider a multiuser wireless system with n mobile users and a base station. We assume slow fading channels where packet transmission error (outage) is the primary concern even if strong channel coding is applied. To capture the effect of potential packet error, we consider the average packet error probability and the total system goodput, which measures the average b/s/Hz successfully delivered to the base station, of the n users. Unlike previous works, our analysis focus on the *error-propagation effects* in MUD-SIC detector where the packet outage event for the i-th decoded user is coupled with that in the $i-1,..,1$-th decoding attempts. We shall derive the optimal SIC decoding order (to maximize system goodput) and evaluate the closed-form *per-user* packet outage probabilities for the n users for MUD-SIC. Simulation results are used to verify the analytical expressions.**

***Index Terms*— Multi-access channel, slow fading, packet error probability, goodput.**

I. INTRODUCTION

THe uplink of wireless cellular system, where many mobile users communicate to a single base station, can be modeled by the multi-access channel. The multi-access channel is characterized by a capacity region, which is the set of achievable rate vector[1] and multi-user detection with successive interference cancellation (MUD-SIC) is one receiver scheme that can achieve the corner points in the dominant face of the multiaccess capacity region[1]. Most of the existing works on multiaccess channel are either focused on signal processing algorithms or performance analysis for multi-user detection. In [2], signal design for multiaccess channel is discussed. In [3], multiuser detection algorithm for overloaded CDMA system is discussed. Conventional performance analysis of multi-access fading channel is usually based on the ergodic capacity[4], [5]. Uplink power adaptation for multiaccess channel is addressed in [6] where the transmit power of mobile users are optimized with respect to a system objective function of user capacities. In all these works, *ergodic capacity* is the key performance measure and optimization objective. However, the ergodic capacity is a reasonable performance measure only for fast ergodic fading channels where a transmitted packet spans across ergodic realizations of channel fading. In this case, the transmitted packets from the mobile users can be guaranteed to be successfully received by the base station as long as powerful channel coding such as LDPC code[7] with sufficiently long block length is applied and the transmitted data rate is less than the ergodic channel capacity. However, for slow fading channels (non-ergodic channels), in which the channel fading is quasi-static within the entire encoding frame, the transmitted packets cannot be guaranteed to be always successfully received even if powerful channel coding is applied. In this case, the instantaneous mutual information of the channel appears as a random variable to the transmitters. The packet transmitted will be corrupted if the data rate is larger than the instantaneous mutual information (despite the use of error correction code) and this is called *packet outage*. Hence, in slow fading channels, ergodic capacity is no longer a useful performance measure because it does not take into account potential packet outage. To include the effects of potential packet errors due to channel outage, we should analyze the *packet outage probability* and *system goodput* (which is defined as the average bits/sec/Hz successfully delivered to the receiver).

In this paper, our focus is to evaluate the *per-user* packet error (outage) probabilities and the system goodput for multi-access slow fading channel with adaptive MUD-SIC. We consider a system with a base station and n mobile users where there is no channel state knowledge (CSIT) at the transmitters of the mobiles. We assume adaptive successive interference cancellation (MUD-SIC) processing at the base station where the decoding order among the n mobile users is adaptive based on the channel state information at the base station (CSIR) so as to maximize the total system goodput. In [8], [6], [9], the delay-limited capacity of the multi-access channel with perfect CSIT is analyzed without considering packet outage events. In [10], [11], the authors analyzed the system goodput for multiaccess channels with optimal (maximal likelihood (ML)) multiuser detection and linear multiuser detection (MMSE). Yet, the results from these works cannot be applied in our case

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[1]In general, joint detection is needed to achieve the multi-access capacity region.

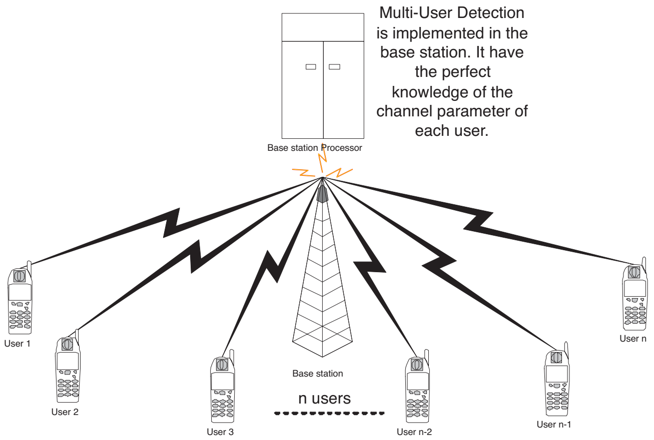

Fig. 1. System model of multi-user network with multi-user detection

with MUD-SIC in quasi-static fading channels. In the case of ML detector (joint detection), the outage event is defined as the event that the rate vector is outside the *instantaneous capacity region*[2] of the multiaccess channel and there is no "notion" of *per-user packet outage* or *error-propagation effects*. In the case of MMSE detectors, the outage event is completely decoupled among the n users. However, when we consider MUD-SIC detector, there is *mutual coupling* (error propagation) of the packet error events between the n users in the SIC decoding process. For example, the packet error event of the user decoded in the k-th iteration depends not only on the packet transmitted by user k but also on all the users decoded in the $(k-1)$-th, $(k-2)$-th,..., and the first round. Furthermore, as illustrated in figure 2, the *per-user packet outage event* (for user 1) cannot be deduced from whether the rate pair is inside or outside the *instantaneous capacity region*. For the rate vector $\vec{r}_A$ (outside the instantaneous capacity region), packet from user 1 can still be successfully decoded if the right decoding order is used. In addition, the choice of decoding order is also very important to the overall system goodput. For rate vector $\vec{r}_B$ in Figure 2 (inside the instantaneous capacity region), if user 1 is decoded first, both packets from user 1 and user 2 will be corrupted and we will have zero system goodput. On the other hand, if user 2 is decoded first, both packets can be successfully received. Furthermore, the packet outage event of user 1 depends not only on the channel state of user 1 but also on that of user 2 as well due to the coupling of the adaptive MUD-SIC. As far as we are aware, the issues of coupled *per-user packet outage events* or error-propagation for MUD-SIC detection have not been addressed previously. In this paper, we shall address two important issues associated with MUD-SIC detection in quasi-static multi-access fading channels where we have homogeneous users with equal data rate[3].

II. System Model

In this section, we shall elaborate on the overall system models and the base station processing for the multi-access

[2]Instantaneous capacity region refers to the multiaccess capacity region for a given channel fading realization.

[3]Equal data rate represents an important class of voice applications in wireless networks

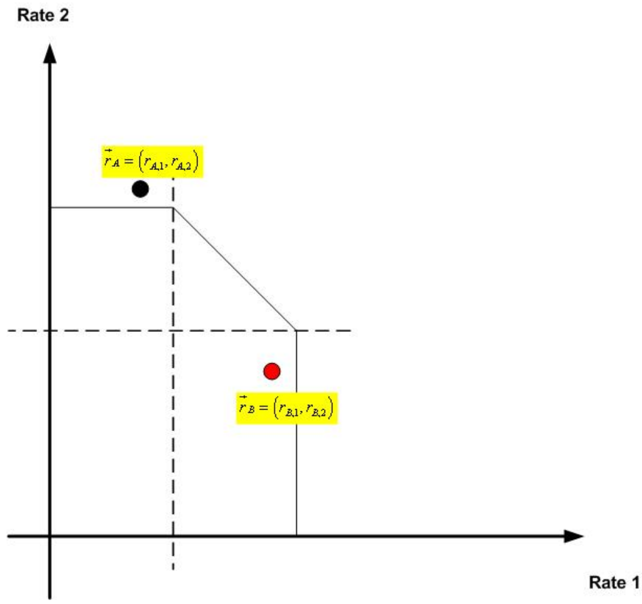

Fig. 2. Illustration of the mutual coupling (error propagation) of packet outage events and the importance of decoding order in MUD-SIC for system goodput considerations in quasi-static multiaccess fading channels. Rate vector $\vec{r}_A$, which is outside the *instantaneous capacity region*, may contribute to non-zero system goodput if user 1 is decoded first. Rate vector $\vec{r}_B$, which is inside the instantaneous capacity region, may contribute to zero system goodput if a wrong decoding order is used.

fading channel. In this paper, capital letter represents random variable and small letter represents a realization of the random variable. $\mathcal{E}[X]$ denotes the expectation of the random variable X. π denotes a decoding order where $\pi(i)$ gives the user index in the i-th decoding iteration and $\pi^{-1}(k)$ gives the decoding order of the k-th user. X^* denotes the complex conjugate of a random variable X and $X_{[i]}$ represents the i-th ordered statistics in a sample size of n.

A. Multi-access Channel Model

Figure 1 illustrates the overall system model. We have a base station and n mobile users. The uplink transmissions of the n mobiles are synchronous so that successive interference cancellation (SIC) is applied at the base station with perfect channel state information (CSIR). On the other hand, the mobile transmitters do not have any channel state information (CSIT). We consider slow flat fading multi-access channels where the channel fading remains quasi-static within the entire transmitted packet. This is a realistic assumption for pedestrian mobility (10km/hr) in most systems such as HSDPA, 3G1X and WiFi, where the coherence time is around 20ms and the frame duration is less than 2ms.

Let X_i be the transmitted symbol from the i-th user with average transmit SNR $E[|X_i|^2] = \sigma_i^2$ and H_i be the channel fading coefficient between the $i-$th mobile and the base station (which is modelled as zero-mean complex Gaussian random variable with covariance $\mathcal{E}[|H_i|^2] = 1$) and $\mathbf{H} = [H_1, ..., H_n]$ be the aggregate channel fading. The received signal at the base station is given by

$$Y = \sum_{i=1}^{n} H_i X_i + Z \tag{1}$$

where Z denotes channel noise, which is modelled as zero mean complex Gaussian with normalized variance $\mathcal{E}[|Z|^2] = 1$.

B. MUD-SIC Processing and Per-User Packet Error Model

The base station has to detect the signal transmitted by the n users based on the received signals Y. In this paper, we assume the base station is equipped with synchronous multi-user detection with successive interference cancellation (MUD-SIC) and perfect channel state information (CSIR). Given a particular decoding order $\pi = (\pi(1), ..., \pi(n))$ where $\pi(i)$ is the user index of the user decoded in the i-th decoding iteration, the instantaneous mutual information (using Gaussian random codebook) of the $\pi(i)-$th user is given by

$$C_{\pi(i)}(\mathbf{H}, \pi, i) = \log_2 \left(1 + \frac{|H_{\pi(i)}|^2 \sigma^2_{\pi(i)}}{\sum_{p=i+1}^{n} (|H_{\pi(p)}|^2 \sigma^2_{\pi(p)}) + 1} \right) \quad (2)$$

where σ_k^2 is the transmit SNR of the $k-$th user.

In this paper, we consider a homogeneous system where the n users transmit information with the same data rate ($r_1 = ... = r_n = r$). This represents an important case such as voice applications in the cellular systems. Since the channel fading is quasi-static within the transmitted packets and the mobile transmitters do not have knowledge of the channel states $H_1, .., H_n$, the instantaneous mutual information of the n users $\{C_1, ..., C_n\}$ appears as random variables to the mobile transmitters. The transmitted packet of the $\pi(i)$-th user will be corrupted if the data rate of the transmitted packet r exceeds the instantaneous mutual information $C_{\pi(i)}$ of individual users. This refers to *packet outage*. In fact, the packet error probability is contributed by two factors, namely the *packet outage* and the *channel noise*. The second factor is due to the finite block length effect of channel coding. Suppose strong enough channel coding is applied and the channel coherence time is much longer than the symbol duration (so that sufficiently long block length can be used), the packet outage will be the dominant factor that contributes to packet error. Note that the packet outage is due to the slow fading channels (non-ergodic channels) and cannot be eliminated even if capacity achieving codes are used at the transmitter. Hence, we shall assume packet error probability is mainly due to the packet outage only and shall use the two words interchangably in the paper.

In this paper, we consider multiaccess channel with adaptive SIC and hence, the decoding order π is a function of the CSIR $\mathbf{H}$. Let $\mathcal{P} = \{\pi_{\mathbf{H}}\}$ denotes the *decoding order policy*, which is a set of decoding order $\pi_{\mathbf{H}}$ with respect to every realization of CSIR $\mathbf{H}$. Given a decoding order policy $\mathcal{P}$, we are interested to find the average PER (averaged over ergodic realization of CSI) of the user k, $\overline{P_{out}}(r, \mathcal{P}, k)$. However, the analysis of per-user PER is not trivial due to the coupling of decoding events in SIC. For example, when user k is decoded in the 3-rd iteration, the success of packet delivery depends not only on the instantaneous mutual information of user k but also on the success/failure in the 1st and 2nd decoding iterations. In fact, the success or failure of a packet transmission of a user cannot be simply told from whether the rate vector is inside the multiaccess capacity region. As illustrated in Figure 2, rate vector $\vec{r}_A$, which is outside the capacity region, may contribute to non-zero system goodput if the correct decoding order is used. To take care of the intrinsic coupling of the adaptive SIC in the PER analysis, we define the *effective instantaneous mutual information*[4] of user $k = \pi(i)$ in the i-th decoding iteration as:

$$\widetilde{C_k}(\mathbf{H}, \pi, i) = \log_2 \left(\frac{1 + \sum_{j=i}^{n} \sigma^2_{\pi(j)} |H_{\pi(j)}|^2 + \widetilde{W_i^\pi}}{1 + \sum_{j=i+1}^{n} \sigma^2_{\pi(j)} |H_{\pi(j)}|^2 + \widetilde{W_i^\pi}} \right) \quad (3)$$

where $\widetilde{W_i}$ denotes the accumulated *undecodable interference* after $i-1$ decoding iterations and it is given by:

$$\widetilde{W_i^\pi} = \sum_{j=1}^{i-1} \sigma^2_{\pi(j)} \left| H_{\pi(j)} \right|^2 \mathcal{I}[r \geq \widetilde{C_{\pi(j)}}(\mathbf{H}, \pi, j)] \quad (4)$$

where $\mathcal{I}[.]$ represents the indicator function[5], r is the transmitted data rate and $\widetilde{W_1^\pi} = 0$.

Hence, the average PER of the user k (averaged over CSIR) is given by:

$$\begin{aligned} & \overline{PER}(r, \mathcal{P}, k) \approx \overline{P_{out}}(r, \mathcal{P}, k) \\ = \; & 1 - \sum_{\pi \in \mathcal{P}} \Pr\left[r < \widetilde{C_k}\left(\mathbf{H}, \pi, \pi^{-1}(k)\right) | \pi \right] \Pr[\pi] \quad (5) \end{aligned}$$

In order to capture the effect of potential packet error due to channel outage, we define the *average system goodput* under a given decoding order policy ($\mathcal{P}$), $\rho(r, \mathcal{P})$, to be the total bits/sec/Hz that successfully delivered to the base station. That is,

$$\overline{\rho}(r, \mathcal{P}) = \sum_{k=1}^{n} r \{1 - P_{out}(r, \mathcal{P}, k)\} \quad (6)$$

Note that both system goodput and the PER are functions of the decoding order policy $\mathcal{P}$. In the next section, we shall deduce the optimal decoding order policy to maximize the average system goodput.

C. Optimal Decoding Order Policy

Note that existing literature that discusses about the optimal decoding order are all based on some utility functions of ergodic capacity [6], [8] in which potential packet errors (outage) of the n users are not taken into consideration. In this section, we shall derive the optimal decoding order (per fading slot) to maximize the system goodput $\rho(r, \mathcal{P})$ as defined in (6). The results are summarized in the lemma below.

Lemma 1 (Optimal Decoding Order)**.** *Given the instantaneous receive SNR $\{\gamma_1, .., \gamma_n\}$ where $\gamma_k = \sigma_k^2 |H_k|^2$, the optimal decoding order that maximize the system goodput $\rho(r, \mathcal{P})$ is given by*

$$\pi(j) = \arg\max_{k \in (S \setminus T)} \left(\gamma_k \right) \quad (7)$$

[4]The effective mutual information here is different from the mutual information in (2) in the sense that the success/failure events in the $i-1, i-2, ..., 1$ decoding attempts are taken care.

[5]$\mathcal{I}[A] = 1$ if the event A is true and zero otherwise.

where $S = \{1..n\}, T = \{\pi(1), .., \pi(j-1)\}$ and the pdf of γ_k is given by

$$f_{\gamma_k}(x_k) = \frac{1}{\sigma_k^2} e^{-\frac{x_k}{\sigma_k^2}} \tag{8}$$

Proof 1. *Please refer to Appendix A.*

Define $\xi_i \in \{0, 1\}$ as the event that the i-th decoded user is decoded successfully ($\xi_i = 1$ denotes successful decoding and $\xi_i = 0$ denotes decoding failure). The event ξ_i is given by (9), where $\mathcal{I}(A)$ is the *indicator function*. Define

$$I_i = \mathcal{I}\left\{ r < \log\left(1 + \frac{\gamma_{\pi(i)}}{1 + \sum_{j>i} \gamma_{\pi(j)}}\right)\right\}. \tag{10}$$

Given the optimal decoding order policy $\mathcal{P}^*$ in (7) and the associated optimal decoding order π, we have $C_{\pi(i)}(\mathbf{H}, \pi, i) > C_u(\mathbf{H}, \pi, i)$ for all $u \neq \pi(i)$. Hence, for user $\pi(i)$ in the i-th decoding iteration, packet error for user $\pi(i)$ can be declared whenever packet error occurs in any of the $j = 1, 2, .., i$-th decoding iterations. In other words, we have

$$\xi_i = 0 \Rightarrow I_1 \cup I_2 \cup .... \cup I_i = 0 \tag{11}$$

Hence, the average packet outage probability of user k transmitting at a rate r in (5) can be simplified as:

$$\begin{aligned}
&\overline{P_{out}}(r, \mathcal{P}^*, k) \\
= &\sum_{\pi \in \mathcal{P}^*} \left(\sum_{i=1}^{\pi^{-1}(k)} \Pr[\xi_i = 0|\pi] \right) \Pr[\pi] \\
= &\sum_{\pi \in \mathcal{P}^*} \left(\sum_{i=1}^{\pi^{-1}(k)} \Pr[I_1 \cup I_2 \cup .... \cup I_i = 0|\pi] \right) \Pr[\pi] \\
\leq &\sum_{\pi \in \mathcal{P}^*} \left(\sum_{i=1}^{\pi^{-1}(k)} \sum_{j=1}^{i} \Pr[I_j = 0|\pi] \right) \Pr[\pi]
\end{aligned} \tag{12}$$

where the final upper bound is due to union bound and $\Pr[I_j = 0|\pi]$ is the conditional packet outage probability in the j-th iteration under the decoding order π. From (10), $\Pr[I_j = 0|\pi]$ is given by:

$$\Pr(I_j = 0|\pi) = \Pr[r > C_{\pi(i)}(\mathbf{H}, \pi, i)|\pi] \tag{13}$$

and $\Pr[\pi]$ is the probability for the decoding order π in the optimal policy $\mathcal{P}^*$ to be selected in the current time slot and is given by

$$\Pr(\pi) = \Pr(\gamma_{\pi(1)} \geq \gamma_{\pi(2)} \geq \gamma_{\pi(3)} .. \geq \gamma_{\pi(n)}). \tag{14}$$

In other words, the average outage probability is the average of the occurrence of all the packet outage events that happen prior to the current decoding iteration (average over every possible decoding order π in the policy $\mathcal{P}^*$).

Similarly, the average system goodput under the optimal decoding order policy $\mathcal{P}^*$ is given by:

$$\begin{aligned}
&\overline{\rho}(r, \mathcal{P}^*) \\
= &\sum_{\pi \in \mathcal{P}^*} \left(\sum_{i=1}^{n} r \Pr[\xi_i = 1|\pi] \right) \Pr(\pi) \\
= &\sum_{\pi \in \mathcal{P}^*} \left(\sum_{i=1}^{n} r \left(1 - \Pr[I_1 \cup I_2 \cup .... \cup I_i = 0|\pi]\right) \right) \Pr(\pi) \\
\geq &\sum_{\pi \in \mathcal{P}^*} \left(\sum_{i=1}^{n} r \left(1 - \sum_{j=1}^{i} \Pr[I_j = 0|\pi]\right) \right) \Pr(\pi)
\end{aligned} \tag{15}$$

III. Performance Analysis

In this section, we shall derive the analytical expressions on the *per-user packet outage probability* and the *system goodput* for MUD-SIC detector under the optimal decoding order policy $\mathcal{P}^*$.

A. System Goodput and Per-User Packet Outage Probability for MUD-SIC

From equations (15) and (12), both the average system goodput $\overline{\rho}$ and the average packet error probability of user k, $\overline{P_{out}}(r, \mathcal{P}^*, k)$, are determined by averaging over all the possible decoding order π under the optimal policy $\mathcal{P}^*$. To obtain the analytical expressions of the average system goodput and the average packet error probability, we have to determine the conditional outage probability of the j-th iteration $\Pr(I_j = 0|\pi)$ in (13) and the probability of choosing the decoding order $\Pr(\pi)$. Given a decoding order π and from (2), $\Pr(I_j = 0|\pi)$ can be expressed as

$$\begin{aligned}
\Pr(I_j = 0|\pi) &= 1 - \Pr[r \leq C_{\pi(j)}(\mathbf{H}, \pi, j)|\pi] \\
&= 1 - \Pr\left[\gamma_{\pi(j)} - \eta \sum_{p=j+1}^{n} \gamma_{\pi(p)} \geq \eta \middle| \pi\right]
\end{aligned} \tag{16}$$

where $\eta = 2^r - 1$. Without loss of generality, we consider a decoding order $\pi = (1, 2, .., n)$. From $\mathcal{P}^*$, the optimal decoding order is in descending order of γ_i. Hence, conditional on $\pi = (1, 2, ..., n)$, we have $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n$. By [12], the joint pdf of the γ_j is then given by (17), where $\Pr(\pi) = \Pr(\gamma_1 \geq \gamma_2 \geq \gamma_3 .. \geq \gamma_n))$. Hence, from the above equations, the computation of $\Pr(I_j = 0|\pi)$ and $\Pr[\pi]$ involved multi-dimensional nested integrals which are cumbersome and complicated. To obtain a tractable analytical expression for the average system goodput, we shall derive the following lemma.

Lemma 2. *The ordered channel gains ($\gamma_{[1]} \geq \gamma_{[2]} \geq ... \geq \gamma_{[n]}$) can be transformed into independent (but not necessarily identical) exponential random variables $\{Z_1, ..., Z_n\}$ by the following transformation*

$$Z_i = i[\gamma_{[i]} - \gamma_{[i+1]}] \tag{18}$$

$$\xi_i = \mathcal{I}\left\{ r < \log\left(1 + \frac{\gamma_{\pi(i)}}{1 + \sum_{j<i}\gamma_{\pi(j)}(1-\xi_j) + \sum_{j>i}\gamma_{\pi(j)}}\right)\right\} \tag{9}$$

$$f_{\gamma_1,\gamma_2,\ldots\gamma_n}(x_1, x_2 \ldots x_n|\pi) = \begin{cases} \frac{1}{\Pr(\pi)\prod_{i=1}^{n}\sigma_i^2}\prod_{i=1}^{n} e^{\frac{-x_i}{\sigma_i^2}} & \text{if } 0 \le x_n \le \ldots \le x_1 < \infty, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where $0 \le Z_i < \infty$ for all $i \in \{1, 2, .., n\}$ and $\gamma_{[n+1]} = 0$. $\{Z_i\}$ is a set of independent exponential random variables with p.d.f. given by:

$$f_{Z_i}(z) = \phi_i e^{-z\phi_i}$$

where the parameter ϕ_i given by

$$\phi_i = \frac{\sum_{u=1}^{i}\sigma_u^{-2}}{i} \tag{19}$$

Proof 2. *Please refer to Appendix B.*

The implication of the above lemma is that the original ordered random variables $\{\gamma_{[i]}\}$ can be transformed[6] into a set of "virtual user" statistics (Z_v) which is independent. By making use of this lemma, the joint pdf of Z_v is then given by

$$f_{z_1,z_2,\ldots z_n}(z_1, z_2 \ldots z_n|\pi) = \frac{1}{n!}\frac{1}{\Pr(\pi)\prod_{i=1}^{n}\sigma_i^2}\prod_{i=1}^{n} e^{-z_i\phi_i} \tag{20}$$

where ϕ_i is given by the equation(19). Hence, from (16), the conditional outage probability $\Pr(I_j = 0|\pi)$ in $j-th$ iteration conditioned on a given decoding order π can be expressed as

$$\begin{aligned} \Pr(I_j = 0|\pi) &= 1 - \Pr\left[\sum_{v=j}^{n}\lambda_v Z_v \ge \eta \middle| \pi\right] \\ &= 1 - \Pr\left[\Gamma_j \ge \eta|\pi\right] \end{aligned} \tag{21}$$

where $\eta = 2^r - 1$, $\Gamma_j = \sum_{v=j}^{n}\lambda_v Z_v$ is a linear combination of $(n-j+1)$ independent exponential random variables ($\{Z_v\}$) and $\lambda_v = \frac{1-(v-j)\eta}{v}$. Now, the conditional outage probability is expressed in terms of a single random variable Γ_j and nested multi-dimensional integration can be avoided.

Making use of the characteristic function of the exponential random variable and the partial fraction theorem, the p.d.f. of the random variable Γ_j is found and summarized in the following lemma.

Lemma 3. *The p.d.f. of Γ_j is given by*

$$f_{\Gamma_j}(x) = \sum_{v=j}^{n}\frac{A_v}{|\overline{\lambda_v}|}B_v \tag{22}$$

[6]Yet, unlike the standard ordered-statistics transformation[12], the transformed variables $\{Z_v\}$ are independent but not necessarily identical due to potentially different transmit SNR σ_i^2 among the n users.

where $x_j \in \Re$ and

$$\begin{aligned} \overline{\lambda_v} &= \lambda_v/\phi_v \quad , \quad A_v = \left(\prod_{u=j,u\ne v}^{n}\frac{\overline{\lambda_v}}{(\overline{\lambda_v} - \overline{\lambda_u})}\right) \\ B_v &= \begin{cases} e^{\frac{-x}{|\overline{\lambda_v}|}}u(x) & \textit{if } \overline{\lambda_v} \ge 0 \\ e^{\frac{x}{|\overline{\lambda_v}|}}u(-x) & \textit{otherwise} \end{cases} \end{aligned} \tag{23}$$

where ϕ_v is given by the equation(19) and $u(x)$ is the unit step function.

Proof 3. *Please refer to Appendix C.*

From above lemma, $\Pr(I_j = 0|\pi)$ is found to be

$$\begin{aligned} \Pr(I_j = 0|\pi) &= 1 - \int_{\eta}^{\infty} f_{\Gamma_j}(x_j)dx_j \\ &= 1 - \sum_{v=j}^{n} A_v e^{\frac{-\eta}{\overline{\lambda_v}}} I(\overline{\lambda_v} \ge 0) \end{aligned} \tag{24}$$

The indicator function $I(\overline{\lambda_v} \ge 0)$ is due to the integration over the region $0 \le \eta < \infty$.

After obtaining the closed-form expression for $\Pr(I_j = 0|\pi)$, we have to obtain the closed form expression for $\Pr(\pi)$. From the p.d.f. expression in equation (17), the probability of the optimal decoding order π can be derived from the fact that the integration of the joint p.d.f., $f_{z_1,z_2,\ldots z_n}(z_1, z_2 \ldots z_n|\pi)$, over the entire space of $Z_1, .., Z_n$ equals to 1. Hence, $\Pr(\pi)$ is given by

$$\Pr(\pi) = \frac{1}{n!}\prod_{i=1}^{n}\frac{1}{\phi_i\sigma_i^2} \tag{25}$$

Note that the probability of optimal decoding order depend on the average received SNR (σ_i^2) of every user. If all user have the same received SNR ($\sigma_1^2 = \sigma_2^2 \ldots \sigma_n^2$), the probability of decoding order become

$$\Pr(\pi) = \frac{1}{n!} \tag{26}$$

Hence, under the special case of equal SNR, every optimal decoding order is statistically equiprobable.

Based on the analytical expressions for $\Pr(I_j = 0|\pi)$ and $\Pr(\pi)$, the average system goodput and the average packet error probability of user k under the optimal decoding order policy $\mathcal{P}^*$ are summarized in the following two theorem.

Theorem 1 (Lower Bound for Average System Goodput of MUD-SIC with Optimal Decoding Order)**.** *The average system goodput $(\overline{\rho})(r, \mathcal{P}^*)$ with optimal decoding order policy $\mathcal{P}^*$ is given by (27), where $\eta = 2^r - 1$.*

$$\overline{\rho}(r,\mathcal{P}^*) \geq \frac{1}{n!}\sum_{\pi\in\mathcal{P}^*}\left[\sum_{i=1}^{n} r\left(1-\sum_{j=1}^{i}\left(1-\sum_{v=j}^{n} A_v e^{\frac{-\eta}{\overline{\lambda_v}}} I(\overline{\lambda_v}\geq 0)\right)\right)\right] \prod_{i=1}^{n}\frac{1}{\phi_i \sigma_{\pi(i)}^2} \tag{27}$$

From the above expression, the first term inside the summation represent the system goodput corresponds to each decoding permutation in the optimal decoding policy. The second term outside the summation correspond to the probability of each permutation inside the decoding policy.

Theorem 2 (Upper Bound for Average Per-User Packet Outage Probability of MUD-SIC with Optimal Decoding Order)**.** *The average packet error probability of user k under the optimal decoding order policy $\mathcal{P}^*$ is given by*

$$\overline{P_{out}}(r,\mathcal{P}^*,k) \leq \frac{1}{n!}\sum_{\pi\in\mathcal{P}^*}\left(\sum_{i=1}^{\pi^{-1}(k)}\sum_{j=1}^{i}\left[1-\sum_{v=j}^{n} A_v e^{\frac{-\eta}{\overline{\lambda_v}}} I(\overline{\lambda_v}\geq 0)\right]\right) \prod_{i=1}^{n}\frac{1}{\phi_i \sigma_{\pi(i)}^2} \tag{28}$$

B. Asymptotic Expressions on Average System Goodput and Per-User Packet Error Probability

In this section, we consider the asymptotic expressions of the average system goodput and the packet error probability under the optimal decoding order policy at large SNRs. Specifically, when the average SNR of all the users $\sigma_1^2 = ... = \sigma_n^2 \to \infty$, the channel capacity of j-th iteration with the optimal decoding order π becomes

$$C_{\pi(j)}(\mathbf{H},\pi,j) = log_2\left(1+\frac{|H_{\pi(j)}|^2}{\sum_{p=j+1}^{n}|H_{\pi(p)}|^2}\right) \tag{29}$$

From the above expression, we observe that the channel capacity $C_{\pi(j)}(\mathbf{H},\pi,j)$ is independent of the average transmit SNRs. Hence, by symmetry, all the decoding order π in $\mathcal{P}^*$ is statistically equiprobable ($\Pr(\pi)=\frac{1}{n!}$). The analytical expression for the system goodput ($\overline{\rho}(r,\mathcal{P}^*)$) under the optimal decoding policy $\mathcal{P}^*$ is given by (30). Similarly, the average packet error probability of user k becomes (31), where $\overline{\lambda_v}$ is given by the equations (19) and (23).

IV. Results and Discussions

In this section, we shall present the numerical results obtained from the analytical expressions and verify them with respect to the simulation results on the average packet error probability and the average system goodput. In the simulation, we consider a single cell uplink wireless communicated system with n single-antenna users. All the channel fading coefficients $\{H_1,...,H_n\}$ are generated as i.i.d. complex Gaussian random realizations with zero mean and unit variance.

To obtain the average system goodput, we count the number of successfully decoded packets for the n users and average it over multiple fading realizations. To obtain the average packet error probability, we count the number of packet errors of a user k and average it over multiple fading realizations. In the simulation, each point of the system goodput and packet error probability are obtained by 20000 fading realizations. We consider two different successive interference cancellation policies, namely the *optimal policy* and the *random policy*.

- **Adaptive SIC with Optimal Decoding Order:** For every CSIR realization, the optimal decoding order is given by the descending order of the user received SNR ($\gamma_i = \sigma_i^2|H_i|^2$). The decoding process stops and all undecoded packets are declared corrupted as soon as there is any packet error in any decoding iteration because the subsequent iterations will surely be failed.
- **SIC with Random Decoding Order:** For every fading realization, a random permutation order is obtained and used as the decoding order. On each iteration, the base station attempts to decode the user using different possible paths as illustrated in Figure 3. Different from the SIC with optimal decoding order, the decoding process continues even there is packet error in the current iteration. This is because in the tree processing as illustrated in Figure 3, there is still a possibility that subsequent decoding iterations will be successful given the current decoding iteration fails. Finally, number of error packets for a user k will be counted and averaged over multiple fading realizations.

Under these two decoding order policies, we would like to compare the performance gains on the average system goodput and average packet error probability. Note that all solid lines represent the theoretical result and dotted markers represent the simulated result. Besides, units of all goodput measurements will be in bits/sec/Hz.

A. Results on the Average System Goodput

Figure 4 shows the average system goodput versus the average user SNR (dB) for $n = 5$ users. Each curve in the graph represent different detection methods with same target average packet error probability (10% and 5%). It can be observed that the system goodput with optimal decoding order increases with SNR but with a diminishing return.

$$\overline{\rho}(r, \mathcal{P}^*) \geq \left[\sum_{i=1}^{n} r \left(1 - \sum_{j=1}^{i} \left(1 - \sum_{v=j}^{n} A_v I(\overline{\lambda_v} \geq 0) \right) \right) \right] \tag{30}$$

$$\overline{P_{out}}(r, \mathcal{P}^*, k) \leq \frac{1}{n!} \sum_{\pi \in \mathcal{P}^*} \sum_{i=1}^{\pi^{-1}(k)} \sum_{j=1}^{i} \left[1 - \sum_{v=i}^{n} A_v I(\overline{\lambda_v} \geq 0) \right] \tag{31}$$

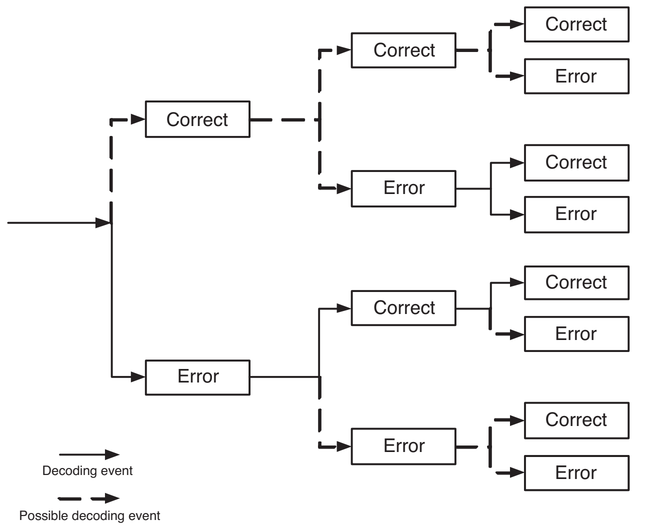

Fig. 3. Illustration of the MUD-SIC decoding tree for random decoding order. The decoding process continues even there is packet error in the current iteration. This is because there is still a possibility that subsequent decoding iterations will be successful given the current decoding iteration fails.

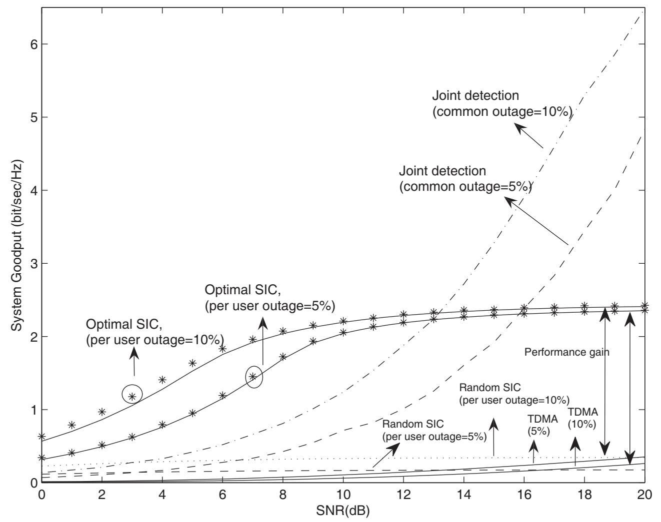

Fig. 4. System goodput vs SNR(dB) with different outage (n=5). The solid line represent the theoretical expression and the dotted solid represent the simulated result of the system goodput respectively. The double sided arrow represent the performance gain of the optimal SIC over the random SIC.

This is because at high SNR, the SNR term in the MUD-SIC will cancel out each other in (2). As a result, the system goodput will be limited by the SINR rather than SNR[7]. Besides, a huge performance gain is found in SIC with the optimal decoding order over the SIC with random decoding order. With random SIC, the average system goodput suffers from the frequent packet decoding error. In order to achieve the same average packet error probability level, each user has to transmit at a lower data rate and this reduces the average system goodput in the case of random decoding order. Furthermore, for the case with random decoding order, the average system goodput does not increase with the SNR[8]. Also, joint ML detection (with *common outage*) is plotted in Figure 4. In low SNR regime, the optimal SIC outperforms the joint ML detection. This is because joint ML detection consider *common outage* and optimal SIC consider *per user outage*. In low SNR regime, the performance of joint ML detection is limited by the decoding error in the weakest user. However, in the optimal SIC, as long as some users can be decoded correctly, it can contribute the system goodput. In high SNR regime, the performance of optimal SIC is limited by strong interference from the other users. On the other hand, the joint ML detection does not suffer from multi-user interference and hence can scale with SNR.

Figure 5 shows the average system goodput versus the number of users n with different average packet error probabilities (10%) . Along all the curves, the same user SNR is fixed at 5dB and 10dB respectively. From the figure, the average system goodput increases as the number of the user increases. Besides, there is also diminishing return when number of users increases. This is because the packet error probability of each user depends on the other users which have been decoded. Similarly, there is a significant performance gain (indicated by the double arrow) between the optimal SIC and the random SIC (except when $n = 1$). As number of the users increases, the importance of the decoding order increases and this contributes to the performance gains. Similar to the above case, the average system goodput in the random SIC case does not scale with number of the users due to the failure of the interferers cancellation. Hence, the optimal decoding order for SIC is very important especially for high SNR cases. Furthermore, joint ML detection method which consider *common outage* is plotted in Figure 5 for comparison. It is very interesting that the system goodput of joint detection (which

[7]Note that for joint detection, the system goodput will not be limited by SINR anymore. Yet, our focus in the paper is to study the performance of MUD-SIC.

[8]The goodput performance under random decoding order is limited not by the SNR but rather by the "packet outage" due to multiuser interference or SINR. For instance, with random decoding order, users decoded at later iterations will most likely suffer from non-zero *accumulated interference* $\widetilde{W}_k^{\pi}$ due to unsuccessful decoding in earlier iterations. Hence, the effective mutual information $\widetilde{C_k}$ is limited by the SINR which saturates at large SNR.

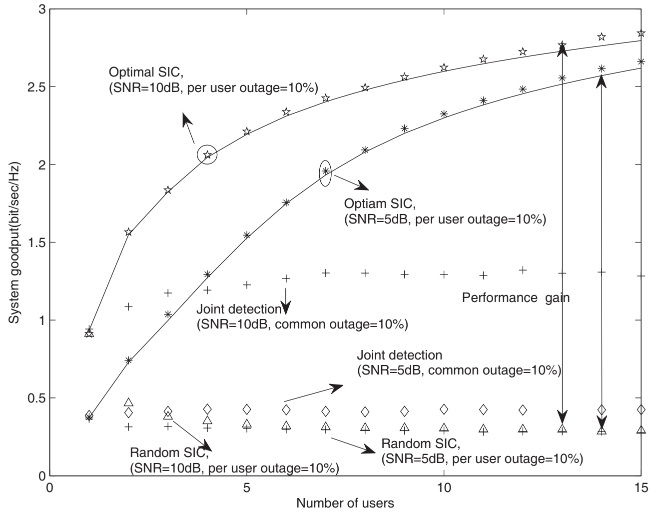

Fig. 5. System goodput against number of users with different SNR (packet error probability=10%)

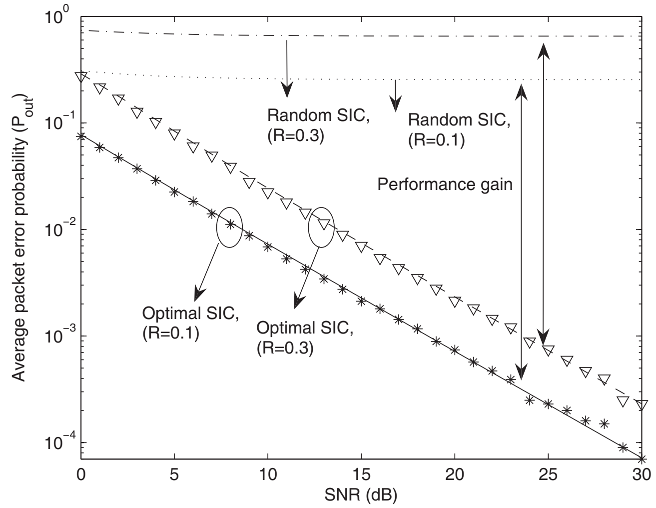

Fig. 6. Average packet error probability against SNR with different transmitted rate(r) (Number of users(n)=5). The solid line represent the theoretical expression and the dotted solid represent the simulated packet error expression respectively. Different curve represent different transmitted rate with the same user. The double sided arrow represent the performance gain of the optimal SIC over the random SIC.

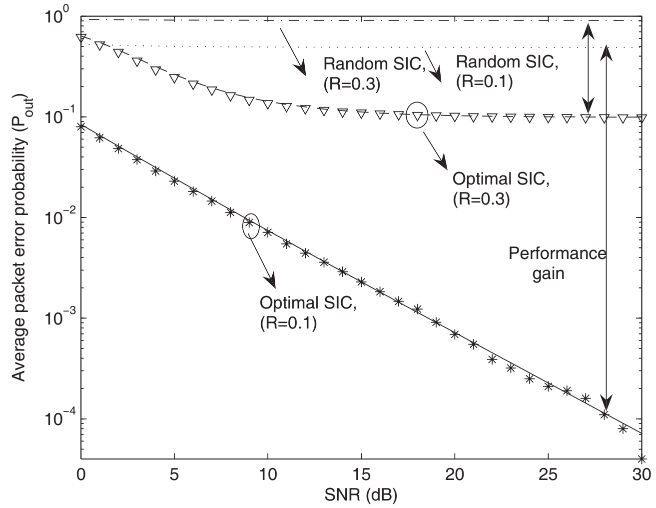

Fig. 7. Average packet error probability against SNR with different transmitted rate(r) (Number of users(n)=10)

consider *common outage*) does not increase with the number of user for a fixed SNR. This counter intuitive result is due to the fact that the performance of joint ML detection (with *common outage*) is always limited by the weakest user. If we don't increase the SNR to help the weakest user, the system goodput cannot scale as the number of users increase.

In all cases, the simulation results match with the analytical results. This verifies the analytical expressions on the average system goodput.

B. Results on Average Per-User Packet Error Probability

Similarly, the average packet error probability versus the average user SNR has been simulated and shown in Figures 6 and 7 for $n = 5$ and $n = 10$ number of users respectively. From the figures, the average packet error probability corresponding to the optimal decoding order policy decreases with the average user SNR. However, with random SIC, the packet error probability does not decrease with increasing SNR.

On the other hand, the average packet error probability versus the number of users at the same average user SNR (5dB and 10dB) and the same transmit data rate R is shown in Figures 8 and 9. It can be observed that with the optimal decoding order policy, the packet error probability increases at a slower rate as the number of users increases. Similarly, in all cases, the simulation results match the analytical results closely, verifying the analytical expressions on the average outage probabilities.

V. Conclusion

In this paper, we have derived the analytical expressions for the average system goodput and the *per-user packet outage* probability for multiacces channel with MUD-SIC. We consider a system with n users and a base station and derive the optimal decoding order at the base station so as to maximize the total average goodput (which measures the b/s/Hz successfully delivered to the base station). Based on the optimal decoding order, we obtain the per-user packet outage probability and the system goodput based on ordered statistics. Numerical result and simulation results are obtained to verify the analytical expressions. The analytical expressions are found to be of close match with the simulations.

Acknowledgment

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Appendix

Appendix A: Proof of Lemma 1

Consider a given CSIR realization $\mathbf{H}$, the optimal decoding order (w.r.t. goodput) is the one that has the largest number of successfully decoded users because the transmit data rate of all the n users are the same. Consider the first iteration, the accumulated *undecodable interference* $\widetilde{W_1^{\pi}} = 0$ and hence,

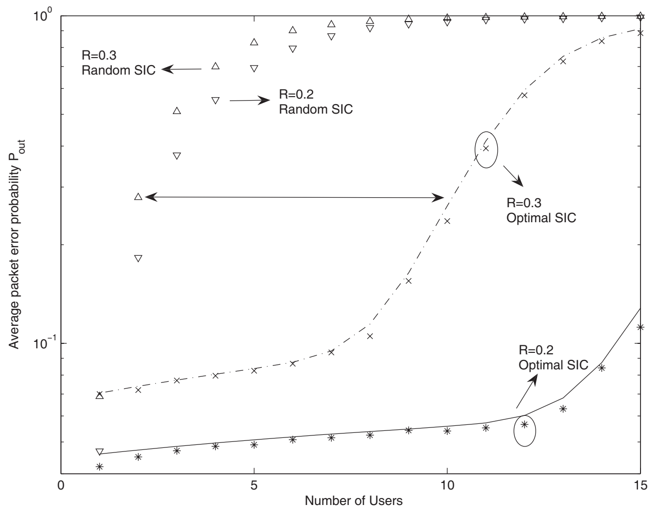

Fig. 8. Average packet error probability against number of users with different transmitted rate(r) (SNR=5dB). The solid line represent the theoretical expression and the dotted solid represent the simulated packet error expression respectively. Different curve represent different transmitted rate with the same SNR(dB). The double sided arrow represent the performance gain of the optimal SIC over the random SIC.

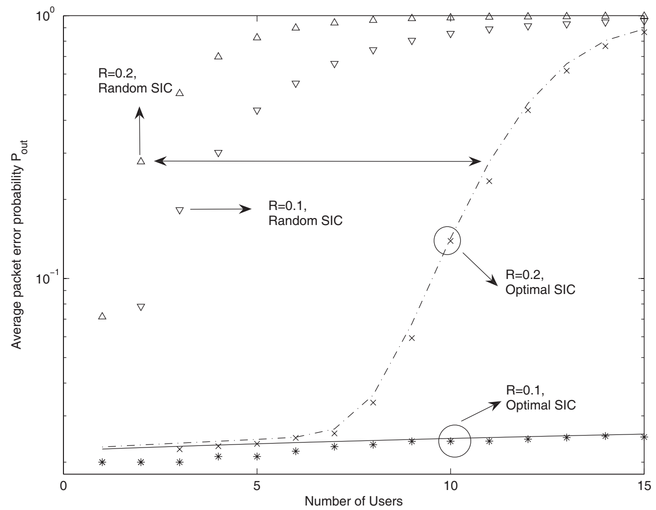

Fig. 9. Average packet error probability against number of users with different transmitted rate(r) (SNR=10dB)

the *effective instantaneous mutual information* in (3) for the first iteration is given by:

$$\widetilde{C_{\pi(1)}}(\mathbf{H},\pi,1)$$
$$= C_{\pi(1)}(\mathbf{H},\pi,1) = \log_2\left(1+\frac{\gamma_{\pi(1)}}{1+\sum_{j=2}^{n}\gamma_{\pi(j)}}\right)$$

Let $\pi^*(1) = \arg\max_{k\in[1,n]}\gamma_k$ be the user with the largest instantaneous SNR and $j \neq \pi^*(1)$ be some other user. If the j-th user can be decoded in the first iteration (i.e. $r < \widetilde{C_j}(\mathbf{H},\pi,1)$), so can the $\pi^*(1)-$th user because $\widetilde{C_j}(\mathbf{H},\pi,1) \leq \widetilde{C_{\pi^*(1)}}(\mathbf{H},\pi,1)$. Hence, we should decode user $\pi^*(1)$ in the first iteration because otherwise, (say decoding user j rather than user $\pi^*(1)$ in the first iteration), such decoding order will result in potentially higher accumulated *undecodable interference* $\widetilde{W_2^\pi} = \gamma_{\pi(1)}\mathcal{I}[r \geq \widetilde{C_{\pi(1)}}(\mathbf{H},\pi,1)]$. As a result, the optimal decoding order (given a CSIR realization) is given by always picking users with the highest SNR. i.e.

$$\pi^*(i) = \arg\max_{k\in[1,n]\setminus\{\pi^*(1),\pi^*(2),..\pi^*(i-1)\}}\gamma_k. \tag{32}$$

APPENDIX B: PROOF OF LEMMA 2

Without loss of generality, consider a particular optimal decoding order $\pi = (1,....,n)$. Hence, we have $\gamma_1 \geq \gamma_2 \cdots \geq \gamma_n$. Applying probability transformation theory for the $1-1$ transformation in (18), the joint pdf of $\{Z_1,...,Z_n\}$ is given by:

$$\begin{aligned}
& f_{Z_1,...,Z_n}(z_1,...,z_n|\pi) \\
= \; & \mathbf{J} \quad f_{\gamma_1,\gamma_2,...\gamma_n}\,(x_1,...,x_n|\pi)|_{x_i=\sum_{j=i}^{n}Z_j/j, i\in\{1,2,..,n\}} \\
= \; & \mathbf{J} \quad \frac{1}{\Pr(\pi)\prod_{i=1}^{n}\sigma_i^2}\prod_{i=1}^{n}e^{-\phi_i z_i}
\end{aligned} \tag{33}$$

where $\mathbf{J}$ denotes the Jacobian of the transformation which is given by $\frac{1}{n!}$ and ϕ_i is defined in (19). From equation (33), the joint p.d.f. can then be expressed as:

$$\begin{aligned}
f_{Z_1,...,Z_n}(z_1,...,z_n|\pi) &= \frac{1}{n!}\frac{1}{\Pr(\pi)\prod_{i=1}^{n}\sigma_i^2}\prod_{i=1}^{n}e^{-\phi_i z_i} \\
&= \prod_{i=1}^{n}G(z_i)
\end{aligned} \tag{34}$$

where $G(z_i) = k_i e^{-\phi_i z_i}$ and k_i can be chosen to satisfy $\int_0^\infty G(z_i) = 1$. Hence, $Z_1, Z_2...Z_n$ are independent (not necessary identical) random variable. By choosing k_i equal to ϕ_i, all Z_i are exponential random variables but with a different parameter ϕ_i which is given by the equation(19).

APPENDIX C: PROOF OF LEMMA 3

The characteristic function of the random variable $\Gamma_m = \sum_{v=m}^{n}\lambda_v Z_v$ is given by:

$$\Phi_{\Gamma_m}(\omega) = \prod_{v=m}^{n}\frac{1}{(1-\overline{\lambda}_v j\omega)} \tag{35}$$

By the partial fraction theorem[13], equation (22) can be expressed as (36). Hence, the characteristic function can be further expressed as the sum of characteristic function of several exponential random variable. After doing the inverse Fourier transform[14], the probability density function is given by:

$$f_{\Gamma_j}(x) = \sum_{v=j}^{n}\frac{A_v}{|\overline{\lambda_v}|}B_v \tag{37}$$

where $x_j \in \Re$ and

$$\begin{aligned}
\overline{\lambda_v} &= \lambda_v/\phi_v \quad , \quad A_v = \left(\prod_{u=j,u\neq v}^{n}\frac{\overline{\lambda_v}}{(\overline{\lambda_v}-\overline{\lambda_u})}\right) \\
B_v &= \begin{cases} e^{\frac{-x}{|\overline{\lambda_v}|}}u(x) & \text{if } \overline{\lambda_v} \geq 0 \\ e^{\frac{x}{|\overline{\lambda_v}|}}u(-x) & \text{otherwise}\end{cases}
\end{aligned} \tag{38}$$

for $x \in \Re$ and the results in the equation (22) follows.

$$\Phi_{\Gamma_m}(\omega) = \sum_{v=m}^{n} \frac{A_v}{(1-\overline{\lambda_v} j\omega)} \quad \text{where} \quad A_v = \prod_{u=m,u\neq v}^{n} \frac{\overline{\lambda_v}}{\overline{\lambda_v}-\overline{\lambda_u}} \tag{36}$$

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