Tomlinson-Harashima Precoding for Multiuser MIMO Systems with Quantized CSI Feedback

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***Abstract*— This paper considers the implementation of Tomlinson-Harashima (TH) precoding for multiuser MIMO systems based on quantized channel state information (CSI) at the transmitter side. Compared with the results in [1], our scheme applies to more general system setting where the number of users in system K can be not equal to the number of transmit antenna n_T ($K \leq n_T$). We also study the achievable average sum rate of the proposed quantized CSI feedback-based TH precoding scheme. The expression of an upper bound on the mean loss in sum rate due to CSI quantization is derived. We also present some numerical results. Both the analytical and numerical results show that nonlinear precoding suffers from imperfect CSI more greatly than linear precoding. Nonlinear TH precoding can achieve much better performance than that of linear zero-forcing precoding for both perfect CSI and quantized CSI cases. In addition, our derived upper bound for TH precoding converges to the true rate loss faster than the upper bound for zero-forcing precoding obtained in [2] as the number of feedback bits increases.**

I. Introduction

The performance of a multiple-input multiple-output (MIMO) communication system with spatial multiplexing is severely impaired by the multi-stream interference due to the simultaneous transmission of parallel data streams. Precoding matches the transmission to the channel and can reduce the inter-stream interference. Accordingly, linear precoding scheme with low complexity are based on zero-forcing (ZF) [3] or minimum mean-square-error (MMSE) criteria [4]. The nonlinear pre-processing based on Tomlinson-Harashima (TH) precoding can achieve significantly better performance than the linear preprocessing algorithm, since it limits the transmitted power increase while pre-eliminating the inter-stream interference [1, 5]. This technique employs modulo arithmetic and has a complexity comparable to that of linear precoders.

As many precoding schemes, the major problem for systems with TH precoding is the availability of the channel state information (CSI) at the transmitter. In time division duplex systems, since the channel can be assumed to be reciprocal, the CSI can be easily obtained from the channel estimation during reception. In frequency division duplex (FDD) systems, the transmitter cannot estimate this information and the CSI has to be communicated from the receivers to the transmitter via a feedback channel. In this paper, we will focus on the implementation of TH precoding in FDD systems. For linear precoding, there have been extensive research results for MIMO systems with quantized CSI at the transmitter [2, 6]. However, as far as we know, there has been very few work for systems employing TH precoding based on quantized CSI at transmit side except the very recent work [7], in which TH precoding was designed based on the available statistics of the channel magnitude information (CMI) and the quantized channel direction information (CDI).

In this paper, we design a multiuser spatial TH precoding based on quantized CSI and ZF criteria. In contrast to [1] where perfect CSI is at the transmitter side, we assume only quantized CDI is available at the transmitter. The feedforward filter as well as the feedback filter are computed at the transmitter only based on the quantized CDI received at the transmitter side. We study the achievable average sum rate of the proposed quantized CSI feedback-based TH precoding scheme by analytically characterizing the average sum rate loss as a function of feedback bits per user. Our derived upper bound for TH precoding tracks the true rate loss quite closely, and appears to converge faster than the upper bound for ZF precoding obtained in [2] as the number of feedback bits increases.

II. System Model

We consider downlink multi-user MIMO systems where TH precoding [8] is used at the transmitter for multi-user interference pre-subtraction. The transmitter is equipped with n_T transmit antennas and K decentralized users each has a single antenna such that $K \leq n_T$. Let the vector $\mathbf{s} = [s_1, \cdots, s_K] \in \mathbb{C}^K$ represent the modulated signal vector, where s_k is the k-th modulated symbol stream for user k. Here we assume that in each of the parallel data streams an M-ary square constellation (M is a square number) is employed and the constellation set is $\mathcal{A} = \{s_I + js_Q |\, s_I, s_Q \in \pm 1\sqrt{\frac{3}{2(M-1)}}, \pm 3\sqrt{\frac{3}{2(M-1)}}, \cdots, \pm\left(\sqrt{M}-1\right)\sqrt{\frac{3}{2(M-1)}}\}$. In general, the average transmit symbol energy is normalized, i.e. $\mathbb{E}\{|s_k|^2\} = 1$. $\mathbf{s}$ is fed to the precoding unit, which consists of a backward square matrix $\mathbf{B}$ and a nonlinear operator $\mathrm{MOD}_\tau(\cdot)$ that acts independently over the real and imaginary parts of its input as follows

$$\mathrm{MOD}_\tau(x) = x - \tau \left\lfloor \frac{x+\tau}{2\tau} \right\rfloor, \quad (1)$$

where $\tau = \sqrt{M}\sqrt{\frac{3}{2(M-1)}}$, $\lfloor z \rfloor$ is the largest integer not exceeding z. $\mathbf{B}$ must be strictly lower triangular to allow data precoding in a recursive fashion [1]. TH precoding modulo (1)

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reduces the transmit symbols into the boundary square region of $\mathcal{R} = \{x + jy | x, y \in (-\tau, \tau)\}$. The channel symbols are equivalently given as

$$x_k = s_k + d_k - \sum_{l=1}^{k-1} [\mathbf{B}]_{k,l} x_l, \tag{2}$$

where $d_k \in \{2\tau(p_I + jp_Q)|\ p_I, p_Q \in \mathbb{Z}\}$ is properly selected to ensure the real and imaginary parts of x_k are constrained into $\mathcal{R}$ [1]. The constellation of the modified data symbols $v_k = s_k + d_k$ is simply the periodic extension of the original constellation along the real and imaginary axes. Equivalently, the effective data symbols v_k are passed into $\mathbf{B}$, which is implemented by the feedback structure. Thus, we have

$$\mathbf{v} = \mathbf{C}\mathbf{x}, \tag{3}$$

where $\mathbf{v} = [v_1, \cdots, v_K]^T$ and

$$\mathbf{C} = \mathbf{B} + \mathbf{I}. \tag{4}$$

We will make the standard observation that the elements of $\mathbf{x}$ are almost uncorrelated and uniformly distributed over the Voronoi region of the constellation $\mathcal{R}$, and that such a model becomes more precise as n_T increases [5, *Theorem* 3.1]. Furthermore, with $\mathbb{E}\left\{\mathbf{s}\mathbf{s}^H\right\} = \mathbf{I}$, $\mathbf{R}_\mathbf{x}$ can be accurately approximated as $\mathbf{R}_\mathbf{x} = \frac{M}{M-1}\mathbf{I}$ [5]. Moreover, the induced shaping loss by the non-Gaussian signaling leads to the fact that the achievable rate can be up to 1.53 dB from the channel capacity [9]. However, as indicated in [1], the so-called shaping loss can be bridged by higher-dimensional precoding lattices. A scheme named "inflated lattice" precoding has been proved to be capacity-achieving in [10]. Thus, following [1], we will ignore the shaping gap in this work.

A channel spatial pre-equalization are performed at transmit side using a feedforward precoding matrix $\mathbf{F} \in \mathbb{C}^{n_T \times K}$. Throughout this work, we assume equal power allocation to all supported users. Then the received signal can be written as

$$\mathbf{r} = \sqrt{\frac{P}{\Gamma}} \mathbf{H}\mathbf{F}\mathbf{x} + \mathbf{n}, \tag{5}$$

where $\mathbf{H} = \left[\mathbf{h}_1^T, \cdots, \mathbf{h}_K^T\right]^T$ is the compact flat fading channel matrix consisting all the users's channel vectors and $\mathbf{h}_k \in \mathbb{C}^{n_T}$ is the channel from the transmitter to user k[1]. Γ is used for transmit power normalization. $\mathbf{F} \in \mathbb{C}^{n_T \times K}$ satisfies transmit power constraint $\frac{P}{\Gamma}\mathrm{Tr}\{\mathbf{F}\mathbf{R}_\mathbf{x}\mathbf{F}^H\} = \frac{P}{\Gamma}\frac{M}{M-1}\mathrm{Tr}\{\mathbf{F}\mathbf{F}^H\} = P$. We assume $\mathbf{n}$ is the white additive noise at all receivers with the covariance $\mathbf{R}_\mathbf{n} = \mathbf{I}$ without loss of generality. Each receiver compensates for the channel gain by dividing by a factor g_k prior to the modulo operation as follows:

$$\mathbf{y} = \mathbf{G}\left(\sqrt{\frac{P}{\Gamma}} \mathbf{H}\mathbf{F}\mathbf{x} + \mathbf{n}\right), \tag{6}$$

where $\mathbf{G} = \mathrm{diag}\left(g_{1,1}, \cdots, g_{K,K}\right)$.

[1]The ordering of the users' channel vectors in $\mathbf{H}$ will affect the precoding order of the users' information signals and further affect the performance of each user. However, at this stage we assume the user channel vectors are randomly ordered. Thus the TH precoding order of the users is $1, 2, \cdots, K$.

Throughout this work, we assume each receiver can obtain perfect knowledge of his own CSI through channel estimation and feeds back this information to the transmitter with zero delay. In this part, we assume this feedback information is perfect at transmitter. With perfect CSI at the transmitter, let the QR decomposition of the compact channels be $\mathbf{H} = \mathbf{R}\mathbf{Q}$, where $\mathbf{R} = [r_{i,j}] \in \mathbb{C}^{K \times K}$ is a lower left triangular matrix and $\mathbf{Q} \in \mathbb{C}^{K \times n_T}$ is a semi-unitary matrix with orthonormal rows which satisfies $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}$. Then the precoding matrix $\mathbf{F}$ is given as $\mathbf{F} = \mathbf{Q}^H$, the scaling matrix $\mathbf{G}$ is given as $\mathbf{G} = \sqrt{\frac{\Gamma}{P}}\mathbf{\Delta}$ with $\mathbf{\Delta} = \mathrm{diag}\left(r_{1,1}^{-1}, \cdots, r_{K,K}^{-1}\right)$ and the feedback matrix reads $\mathbf{B} = \mathbf{\Delta}\mathbf{H}\mathbf{F} - \mathbf{I} = \mathbf{\Delta}\mathbf{R} - \mathbf{I}$. According to transmit power constraint $\frac{P}{\Gamma}\frac{M}{M-1}K = P$ we have $\Gamma = \frac{M}{M-1}K$. With these processing, the effective data symbols $\mathbf{y}$ corrupted by additive noise can be written as [1]

$$\mathbf{y} = \mathbf{v} + \mathbf{G}\mathbf{n}. \tag{7}$$

At the receivers, each symbol in $\mathbf{y}$ is firstly modulo reduced into the boundary region of the signal constellation $\mathcal{A}$. A quantizer of the original constellation will follow the modulo operation to detect the received signals. The signal-to-noise ratio (SNR) for receiver k ξ_k can be written as

$$\xi_k = \frac{P}{\Gamma}|r_{k,k}|^2. \tag{8}$$

In the following part, we will describe how to implement the precoding with quantized CSI obtained at the transmitter.

III. System with Quantized Transmit CSI

In a FDD system, the transmitter obtains transmit CSI for the downlink through the limited feedback of B bits by each receiver. Following the the studies of quantized CSI feedback in [2, 6], channel direction vector is quantized at each receiver, and the corresponding index is fed back to the transmitter via an error and delay-free feedback channel. Given the quantization codebook $\mathbb{W} = \{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ ($\mathbf{w}_i \in \mathbb{C}^{1 \times n_T}$), which is known to both the transmitter and all receivers, each receiver selects the quantized channel direction vector of its own channel as follows:

$$\hat{\mathbf{h}}_k = \arg\max_{\mathbf{w}_i \in \mathbb{W}} \{|\bar{\mathbf{h}}_k \mathbf{w}_i|^2\}, \tag{9}$$

where $\bar{\mathbf{h}}_k = \frac{\mathbf{h}_k}{\|\mathbf{h}\|}$ is the channel direction vector of user k.

In this work, we use random vector quantization (RVQ) codebook, in which the n quantization vectors are independently and isotropically distributed on the n_T–dimensional complex unit sphere [2]. Although RVQ is suboptimal for a finite-size system, it is very amenable to analysis and also its performance is close to the optimal quantization [2]. Using the result in [2], for user k we have

$$\bar{\mathbf{h}}_k = \hat{\mathbf{h}}_k \cos\theta_k + \tilde{\mathbf{h}}_k \sin\theta_k, \tag{10}$$

where $\cos^2\theta_k = |\bar{\mathbf{h}}_k \hat{\mathbf{h}}_k^H|^2$, $\tilde{\mathbf{h}}_k \in \mathcal{C}^{1 \times M}$ is a unit norm vector isotropically distributed in the orthogonal complement subspace of $\hat{\mathbf{h}}_k$ and independent of $\sin\theta_k$. Then $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{\Gamma}\left(\mathbf{\Phi}\hat{\mathbf{H}} + \mathbf{\Omega}\tilde{\mathbf{H}}\right), \tag{11}$$

where $\mathbf{\Gamma} = \mathrm{diag}\left(\rho_1, \cdots, \rho_K\right)$ with $\rho_k = \|\mathbf{h}_k\|$, $\mathbf{\Phi} = \mathrm{diag}\left(\cos\theta_1, \cdots, \cos\theta_K\right)$ and $\mathbf{\Omega} = \mathrm{diag}\left(\sin\theta_1, \cdots, \sin\theta_K\right)$, $\hat{\mathbf{H}} = \left[\hat{\mathbf{h}}_1^T, \cdots, \hat{\mathbf{h}}_K^T\right]^T$ and $\tilde{\mathbf{H}} = \left[\tilde{\mathbf{h}}_1^T, \cdots, \tilde{\mathbf{h}}_K^T\right]^T$.

With the quantized CDI at the transmitter side, the transmitter obtain the feedforward precoding matrix $\mathbf{F}$ and feedback matrix $\mathbf{B}$ through the QR decomposition of compact channel matrix $\hat{\mathbf{H}}$ in the same way as the QR decomposition of matrix $\mathbf{H}$, i.e. $\hat{\mathbf{H}} = \hat{\mathbf{R}}\hat{\mathbf{Q}}$, where matrices $\hat{\mathbf{R}}$ and $\hat{\mathbf{Q}}$ have the same structure as matrices $\mathbf{R}$ and $\mathbf{Q}$ respectively. In addition, we denote $|\hat{r}_{i,j}|^2$ as the (i, j)-th element of matrix $\hat{\mathbf{R}}$ and $\hat{\mathbf{q}}_l$ as the l-th row of matrix $\hat{\mathbf{Q}}$. Then we have $\mathbf{F} = \hat{\mathbf{Q}}^H$ and $\mathbf{B} = \left(\mathrm{diag}\left\{\hat{\mathbf{R}}\right\}\right)^{-1}\hat{\mathbf{R}} - \mathbf{I}$. In addition, the scaling matrix at the receivers now becomes

$$\mathbf{G} = \sqrt{\frac{\Gamma}{P}}\left(\mathbf{\Gamma}\mathbf{\Phi}\ \mathrm{diag}\left\{\hat{\mathbf{R}}\right\}\right)^{-1}. \tag{12}$$

Using the same operation at the receiver side as that in perfect CSI case to detect the received signals, $\hat{\mathbf{y}}$ can be further written as

$$\begin{aligned}\hat{\mathbf{y}} &= \mathbf{G}\left(\sqrt{\frac{P}{\Gamma}}\mathbf{H}\mathbf{F}\mathbf{x} + \mathbf{n}\right)\\ &= \mathbf{G}\sqrt{\frac{P}{\Gamma}}\mathbf{\Gamma}\left(\mathbf{\Phi}\hat{\mathbf{H}} + \mathbf{\Omega}\tilde{\mathbf{H}}\right)\mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{n}\\ &= \mathbf{v} + \left(\mathbf{\Phi}\ \mathrm{diag}\left\{\hat{\mathbf{R}}\right\}\right)^{-1}\mathbf{\Omega}\tilde{\mathbf{H}}\hat{\mathbf{Q}}^H\mathbf{x}\\ &\quad + \sqrt{\frac{\Gamma}{P}}\left(\mathbf{\Gamma}\mathbf{\Phi}\ \mathrm{diag}\left\{\hat{\mathbf{R}}\right\}\right)^{-1}\mathbf{n}, \end{aligned} \tag{13}$$

where we have used the relationship $\mathbf{v} = \left(\mathrm{diag}\left\{\hat{\mathbf{R}}\right\}\right)^{-1}\hat{\mathbf{R}}\mathbf{x}$. The first term is the useful signal for all users and the second term is interference caused by quantized CSI.

According to (13), the signal-to-interference-plus-noise ratio (SINR) γ_k for receiver k can be written as

$$\begin{aligned}\gamma_k &= \frac{1}{\frac{\sin^2\theta_k}{|\hat{r}_{k,k}|^2\cos^2\theta_k}\|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2 + \frac{\Gamma}{P}\frac{1}{\rho_k^2|\hat{r}_{k,k}|^2\cos^2\theta_k}}\\ &= \frac{\frac{P}{\Gamma}\rho_k^2|\hat{r}_{k,k}|^2\cos^2\theta_k}{\frac{P}{\Gamma}\rho_k^2\|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2\sin^2\theta_k + 1}. \end{aligned} \tag{14}$$

IV. Average Sum Rate Analysis under Quantized CSI Feedback

In this section, we study the achievable average sum rate of the proposed quantized CSI feedback-based TH precoding scheme. To simplify analysis, we will characterize the average sum rate loss as a function of feedback bits per user. For tractability, throughout this section we assume each user's channel is Rayleigh-faded. In the following subsection, we will first study the statistical distribution of the power of interference signal at each user caused by quantized CSI at the transmitter side.

A. Interference Part

In this subsection, assuming Rayleigh fading channel and RVQ for quantized CSI feedback, we will derive the statistical distribution of interference part $\frac{P}{\Gamma}\rho_k^2\|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2\sin^2\theta_k$ in (14). It is well known that ρ_k^2 has distribution $\chi^2_{2n_T}$ and the distribution of $\sin^2\theta_k$ is given in [2, 6]. However, since $\tilde{\mathbf{h}}_k \perp \hat{\mathbf{h}}_k$ $(k = 1, \cdots, K)$ and $\hat{\mathbf{Q}}$ is determined by $\hat{\mathbf{h}}_k$ $(k = 1, \cdots, K)$, $\tilde{\mathbf{h}}_k$ for $k = 1, \cdots, K$ are not independent of $\hat{\mathbf{Q}}$. Thus, the distribution of the term $\|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2$ is still unknown and to obtain the exact result is not trivial. The following lemma presents the exact distribution of this interference term. It is one of the key contributions of this paper.

Lemma 1: For $1 < K < n_T$, the random variables $\varepsilon_k = \|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2$ for $k = 1, \cdots, K$ follow the same beta distribution with shape $(K-1)$ and $(n_T - K)$ which is denoted as $\varepsilon_k \sim \mathrm{Beta}(K-1, n_T - K)$. In addition, the probability density function (p.d.f.) of ε_k is given as

$$f_{\varepsilon_k}(x) = \frac{1}{\beta(K-1, n_T - K)}x^{K-2}(1-x)^{n_T - K - 1}. \tag{15}$$

where $\beta(a, b) = \int_0^1 t^{a-1}t^{b-1}\mathrm{d}t$ is beta function [11]. Specially, when $K = 1$ there is no interference term. When $K = n_T$ $\varepsilon_k = \|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2$ is equal to 1 which is a constant.

Proof: See Appendix I. ■

Lemma 1 implies an very interesting result that, with randomly ordered user channel vectors, the signal of the user which is precoded ahead suffers from the same interference signal power as the signals of the users which are precoded afterwards. In the following we will only focus on the general situation that $1 < K < n_T$. However, it is easy to check that all obtained results also apply to special cases of $K = 1$ and $K = n_T$.

B. An Upper Bound on The Average Sum Rate Loss

The average achievable rate for user k with perfect CSI and quantized CSI feedback are given as

$$R_{P,k} = \log_2\left(1 + \xi_k\right) \text{ and } R_{Q,k} = \log_2(1 + \gamma_k),$$

respectively.

Theorem 1: With B feedback bits per user, the average sum rate loss of user k due to quantized CSI feedback can be upper bounded by[2]

$$\begin{aligned}\Delta R_k &= \mathbb{E}\{R_{P,k} - R_{Q,k}\}\\ &\le \log_2\left(1 + \frac{P}{\Gamma}\frac{n(K-1)n_T}{n_T - 1}\ \beta\left(n, \frac{n_T}{n_T - 1}\right)\right)\\ &\quad + \frac{\log_2(e)}{n_T - 1}\sum_{i=1}^{n_T - 1}\beta\left(n, \frac{i}{n_T - 1}\right), \end{aligned} \tag{16}$$

where $n = 2^B$ is the size of codebook.

The proof is omitted due to limited space. From the results in [2, *Theorem 1*], the sum rate loss due to quantized feeback

[2]Note that, in contrast to ZF precoding, for TH precoding different users have different average sum rate loss. Interestingly, simulation shows that for finite SNR the users precoded earlier will suffer from greater sum rate loss. However, in this work will adopt the average sum rate loss over all supported users.

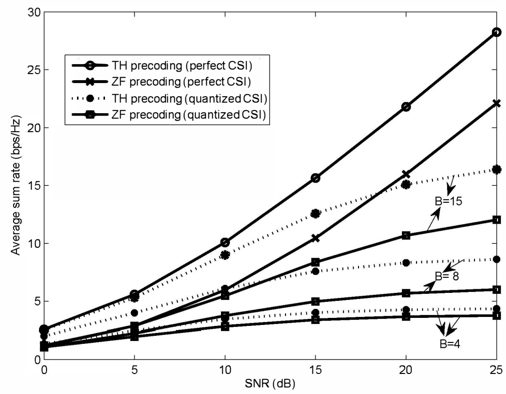

Fig. 1. The average sum rate performance of TH precoding and ZF precoding for both perfect CSI and quantized CSI.

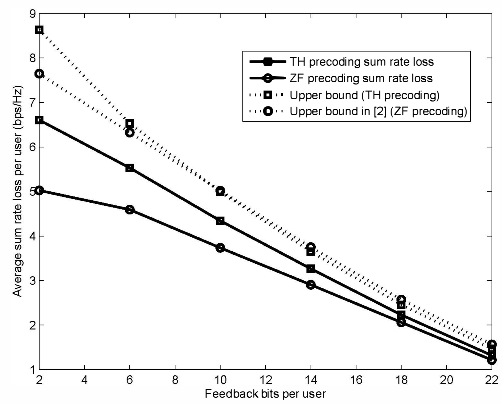

Fig. 2. The average sum rate loss per user and corresponding upper bounds against the number of feedback bits. $P = 25$ dB.

for zero-forcing beamforming is upper bounded by $\Delta R < \log_2\left(1 + Pn\beta\left(n, \frac{n_T}{n_T-1}\right)\right)$. We find the first term at the right hand side (RHS) of (16) is approximately ΔR for high order constellation, large number of transmit antennas and large number of supported users. Thus, the second term at the RHS of (16) can be seen as the sum rate degradation of nonlinear precoding compared with that of linear precoding when only quantized CSI is available at transmit side.

V. Numerical Results

In this section we present some numerical results. We assume $n_T = K = 4$. Here the SNR of the systems is defined to be equal to P.

Fig. 1 shows average sum rate performance of TH precoding and linear ZF precoding for both perfect CSI and quantized CSI with $4, 8$ and 15 feedback bits per user. We can see TH precoding performs better than linear precoding in both perfect CSI and quantized CSI cases. When the SNR is small and moderate, the average sum rate achieved by quantized CSI-based TH precoding can even be better than that of perfect CSI-based linear ZF precoding.

Fig. 2 plots the average sum rate loss per user as a function of the number of feedback bits for both ZF precoding and TH precoding in a system at an SNR of 25 dB. We also plot the upper bound from *Theorem* 1 in this paper and the upper bound from *Theorem* 1 in [2]. From the figure we can see that nonlinear precoding suffers from imperfect CSI more greatly than linear precoding does. However, the performance of nonlinear precoding can still be better than linear precoding when SNR is not large or the feedback quantization resolution is high enough. In addition, we notice that the upper bound for TH precoding tracks the true rate loss quite closely, and appears to converge faster than the upper bound for linear precoding obtained in [2] as B increases.

VI. Conclusion

In this paper, we have investigated the implementation of TH precoding in downlink multiuser MIMO system with quantized CSI at transmit side. In particular, our scheme generalized the results in [1] to more general system setting where the number of users in system K can be not equal to the number of transmit antenna n_T. In addition, we studied the achievable average sum rate of the proposed scheme by deriving an expression of an upper bound on the mean loss in sum rate due to CSI quantization. Our numerical results showed that nonlinear TH precoding could achieve much better performance than that of linear zero-forcing precoding for both perfect CSI and quantized CSI cases. In addition, our derived upper bound for TH precoding converged to the true rate loss faster than the upper bound for zero-forcing precoding obtained in [2] as the number of feedback bits increase.

Appendix I
Proof of *Lemma* 1

The results for the special cases that $K = 1$ and n_T are trivial. In the following we will consider the cases that $1 < K < n_T$. Since the user channel vectors in $\mathbf{H}$ are unordered, so are the quantized channel direction vectors in $\hat{\mathbf{H}} = \left[\hat{\mathbf{h}}_1^T, \cdots, \hat{\mathbf{h}}_K^T\right]^T$. According to QR decomposition of $\hat{\mathbf{H}}$ we have

$$\hat{\mathbf{h}}_k = \sum_{l=1}^{k} \hat{r}_{k,l}\hat{\mathbf{q}}_l, \tag{17}$$

If we require $\hat{r}_{i,i} > 0$ for $i = 1, \cdots, K$, this decomposition is *unique*. Particularly, we have $\hat{r}_{1,1} = 1$ and $\hat{\mathbf{q}}_1 = \hat{\mathbf{h}}_1$. In addition, $\tilde{\mathbf{h}}_k$ are isotropically distributed in the null space of $\hat{\mathbf{h}}_k$ [2]. Thus, for $k = 1$ we have $\tilde{\mathbf{h}}_1 \perp \hat{\mathbf{q}}_1$ or equivalently $\tilde{\mathbf{h}}_1$ is an isotropically distributed unit vector in the null space of $\hat{\mathbf{q}}_1$.

With the assumption of RVQ, the quantized channel direction vectors $\hat{\mathbf{h}}_k, k = 1, \cdots, K$ are independently and isotropically distributed on the n_T–dimensional complex unit sphere due to the assumption of i.i.d. Rayleigh fading. Thus we can conclude that the orthonormal basis $\hat{\mathbf{q}}_1, \cdots, \hat{\mathbf{q}}_K$ of the subspace spanned by quantized channel vectors $\hat{\mathbf{h}}_k, k =$

$$f(t_1,\ldots,t_{K-1}) = \begin{cases} \frac{\Gamma(n_T-1)}{\Gamma(n_T-K)}\left(1-\sum_{i=1}^{K-1} t_i\right)^{n_T-K-1}, & t_i \geq 0, i=1,\cdots,K-1, \sum_{i=1}^{K-1} t_i = 1 \\ 0, & \text{otherwise} \end{cases}. \tag{18}$$

$1,\cdots,K$ have no preference of direction, i.e., $\left[\hat{\mathbf{q}}_1^T,\cdots,\hat{\mathbf{q}}_K^T\right]^T$ is isotropically distributed in the $K \times n_T$ semi-unitary space. Thus, to derive the distribution of $\varepsilon_1 = \|\tilde{\mathbf{h}}_1 \hat{\mathbf{Q}}^H\|^2$, we can assume $\hat{\mathbf{q}}_i = \mathbf{e}_i$ for $i = 1,\cdots,K$ without loss of generality, where $\mathbf{e}_i$ is the i-th row of the identity matrix $\mathbf{I}_{n_T}$. Recall that $\tilde{\mathbf{h}}_1 \perp \hat{\mathbf{q}}_1$, thus the random vector $\tilde{\mathbf{h}}_1$ can be written in the form of $\tilde{\mathbf{h}}_1 = [0,\mathbf{v}]$, where the vector $\mathbf{v} = [v_1, v_2,\cdots,v_{n_T-1}]$ is isotropically distributed on the (n_T-1)–dimensional complex unit sphere. Then $\varepsilon_1 = \|\tilde{\mathbf{h}}_1 \hat{\mathbf{Q}}^H\|^2 = \sum_{l=1}^{K-1} |v_l|^2$. Let $t_l = |v_l|^2$. It has been obtained in [12] that the joint p.d.f. of $t_1,\cdots,t_{K-1}$ is given by (18) which is shown at the top of next page. Now we want to obtain the distribution of $u_1 = \sum_{l=1}^{K-1} t_l$. We define the following transformation of variables

$$u_1 = \sum_{l=1}^{K-1} t_l, \quad u_i = t_i \quad \text{for } i = 2,\cdots,K-1.$$

It is easy to obtain the corresponding Jacobian is $J = 1$. Thus the joint p.d.f. of $u_1,\cdots,u_{K-1}$ is

$$f_{u_1,\ldots,u_{K-1}}(x_1,\cdots,x_{K-1}) = \frac{\Gamma(n_T-1)}{\Gamma(n_T-K)}(1-x_1)^{n_T-K-1}. \tag{19}$$

Since $0 \leq t_i \leq 1$, we have $0 \leq t_1 = u_1 - \sum_{i=2}^{K-1} u_i \leq 1$. The region of the random variables can be obtained as $\mathcal{D} = \left\{(u_1,\cdots,u_{K-1}) \mid 0 \leq \sum_{i=2}^{K-1} u_i \leq u_1 \leq 1, 0 \leq u_i \leq 1 \text{ for } i = 2,\cdots,K-1\right\}$. Then the marginal distribution of u_1 can be obtained as

$$\begin{aligned} &f_{u_1}(x) \\ &= \int\cdots\int_{\mathcal{D}} f(x, x_2\ldots,x_{K-1})\, \mathrm{d}x_2\cdots\mathrm{d}x_{K-1} \\ &= \int\cdots\int_{\mathcal{D}} \frac{\Gamma(n_T-1)}{\Gamma(n_T-K)}(1-x_1)^{n_T-K-1}\, \mathrm{d}x_2\cdots\mathrm{d}x_{K-1} \\ &\overset{(a)}{=} \frac{\Gamma(n_T-1)}{\Gamma(n_T-K)}(1-x)^{n_T-K-1}\frac{x^{K-2}}{(K-2)!} \end{aligned}$$

which is given by (15), where in (a) we have used the identity $\underset{\substack{\sum_{i=1}^n t_i \leq h \\ t_1\geq 0,\cdots,t_n \geq 0}}{\int\int\cdots\int} \mathrm{d}t_1\cdots\,\mathrm{d}t_n = \frac{h^n}{n!}$ [11]. We find that $\varepsilon_1 = u_1$ follows beta distribution with shape $(K-1)$ and (n_T-K). In the following we will prove ε_ks have the same distribution.

We denote $\hat{\mathbf{H}}_{\boldsymbol{\pi}} = \mathbf{P}_{\boldsymbol{\pi}}\hat{\mathbf{H}} = \left[\mathbf{h}_{\boldsymbol{\pi}(1)}^T,\cdots,\mathbf{h}_{\boldsymbol{\pi}(K)}^T\right]^T$ the matrix obtained by permutating the row vector of matrix $\hat{\mathbf{H}}$ according to the permutation $\boldsymbol{\pi}$ of $(1,2,\cdots,K)$, where $\boldsymbol{\pi}$ is channel-independent and $\mathbf{P}_{\boldsymbol{\pi}} = \left[\mathbf{1}_{\boldsymbol{\pi}(1)},\cdots,\mathbf{1}_{\boldsymbol{\pi}(K)}\right]^T$ is the permutation matrix corresponding to $\boldsymbol{\pi}$ and $\mathbf{1}_{\boldsymbol{\pi}(i)}$ is the $\boldsymbol{\pi}(i)$-th column of identity matrix. Then the QR decomposition of $\hat{\mathbf{H}}_{\boldsymbol{\pi}}$ can be written as $\hat{\mathbf{H}}_{\boldsymbol{\pi}} = \mathbf{P}_{\boldsymbol{\pi}}\hat{\mathbf{R}}\hat{\mathbf{Q}} = \hat{\mathbf{R}}_{\boldsymbol{\pi}}\hat{\mathbf{Q}}_{\boldsymbol{\pi}}$. With the assumption that $\hat{\mathbf{R}}_{\boldsymbol{\pi}}$ has positive diagonal elements, the above QR decomposition of $\hat{\mathbf{H}}_{\boldsymbol{\pi}}$ is unique. Using Givens transformation, there is a series of Givens matrices $\mathbf{G}_1,\cdots,\mathbf{G}_{K-1} \in \mathbb{C}^{K\times K}$ which satisfy $\mathbf{P}_{\boldsymbol{\pi}}\hat{\mathbf{R}}\mathbf{G}_1\cdots\mathbf{G}_{K-1} = \bar{\mathbf{R}}_{\boldsymbol{\pi}}$, where $\bar{\mathbf{R}}_{\boldsymbol{\pi}} \in \mathbb{C}^{K\times K}$ is a lower triangular matrix with positive diagonal elements. Since Givens matrix is unitary, we have $\mathbf{G}_1\cdots\mathbf{G}_{K-1}\mathbf{G}_{K-1}^H\cdots\mathbf{G}_1^H = \mathbf{I}$. So $\hat{\mathbf{H}}_{\boldsymbol{\pi}}$ can be written as $\hat{\mathbf{H}}_{\boldsymbol{\pi}} = \bar{\mathbf{R}}_{\boldsymbol{\pi}}\mathbf{G}_{K-1}^H\cdots\mathbf{G}_1^H\hat{\mathbf{Q}}$. Let $\bar{\mathbf{Q}}_{\boldsymbol{\pi}} = \mathbf{G}_{K-1}^H\cdots\mathbf{G}_1^H\hat{\mathbf{Q}}$, we have $\hat{\mathbf{H}}_{\boldsymbol{\pi}} = \bar{\mathbf{R}}_{\boldsymbol{\pi}}\bar{\mathbf{Q}}_{\boldsymbol{\pi}}$ where $\bar{\mathbf{Q}}_{\boldsymbol{\pi}}$ is unitary. Thus $\hat{\mathbf{H}}_{\boldsymbol{\pi}} = \bar{\mathbf{R}}_{\boldsymbol{\pi}}\bar{\mathbf{Q}}_{\boldsymbol{\pi}}$ is also a QR decomposition of $\hat{\mathbf{H}}_{\boldsymbol{\pi}}$. Using the uniqueness of QR decomposition, we conclude that $\bar{\mathbf{Q}}_{\boldsymbol{\pi}} = \hat{\mathbf{Q}}_{\boldsymbol{\pi}}$ and $\bar{\mathbf{R}}_{\boldsymbol{\pi}} = \hat{\mathbf{R}}_{\boldsymbol{\pi}}$. Thus we have

$$\begin{aligned} \varepsilon_k &= \|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}^H\|^2 \\ &= \|\tilde{\mathbf{h}}_k\hat{\mathbf{Q}}_{\boldsymbol{\pi}}^H\mathbf{G}_{K-1}^H\cdots\mathbf{G}_1^H\|^2 \\ &\overset{(b)}{=} \|\tilde{\mathbf{h}}_k\bar{\mathbf{Q}}_{\boldsymbol{\pi}}^H\|^2, \end{aligned} \tag{20}$$

where (b) is due the fact that matrix $\mathbf{G}_i$ is unitary for $i = 1,\cdots,K-1$. If we let $\boldsymbol{\pi}(1) = k$, $\hat{\mathbf{h}}_k$ will be the first row of $\hat{\mathbf{H}}_{\boldsymbol{\pi}}$. According to the previous derivation in the proof, we know ε_k for $k = 2,\cdots,K-1$ have the same distribution as ε_1 whose p.d.f. is give by (15).

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