Iterative Frequency Estimation by Interpolation on Fourier Coefficients

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***Abstract*— The estimation of the frequency of a complex exponential is a problem that is relevant to a large number of fields. In this paper we propose and analyze two new frequency estimators that interpolate on the Fourier coefficients of the received signal samples. The estimators are shown to achieve identical asymptotic performances. They are asymptotically unbiased and normally distributed with a variance that is only 1.0147 times the asymptotic Cramer-Rao bound (ACRB) uniformly over the frequency estimation range.**

***Index Terms*— Digital signal processing, frequency estimation, parameter estimation.**

I. Introduction

In this paper, we consider the estimation of the frequency of a complex exponential, s, given by

$$s(k) = Ae^{j\left[2\pi k\frac{f}{f_s}+\theta\right]} + w(k), k = 0, 1..N-1 \tag{1}$$

where A is the signal amplitude, f the signal frequency and θ the initial phase. N samples are used and the sampling frequency is f_s. The noise terms, $w(k)$, are assumed to be zero mean, complex additive white Gaussian noise with variance σ^2. The signal to noise ratio is given by $\rho = \frac{A^2}{\sigma^2}$. We set, without loss of generality, $A = 1$ and $f_s = 1$. Although the noise is assumed to be white Gaussian, the derivation of the asymptotic properties of the estimators holds under weaker, more general conditions. These relaxed conditions are stated in [1] for the case of real valued noise. However, their extension to the complex case is straightforward and is not explicitly carried out here. The results obtained in this paper are easily extended to the more general case by replacing σ^2 with the power spectral density of the noise at the frequency of interest.

The frequency estimation problem outlined above is relevant to a wide range of areas such as radar, sonar and communications, and has consequently received significant attention in the literature [2] and [3]. It is well known that the maximum likelihood (ML) estimator of the frequency is given by the argument of the periodogram maximizer, [4]. That is,

$$\hat{f}_{ML} = \arg\max_{\lambda}\{Y(\lambda)\} \tag{2a}$$

where

$$Y(\lambda) = \left|\sum_{k=0}^{N-1} s(k)e^{-j2\pi\lambda}\right|^2. \tag{2b}$$

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The Cramer-Rao bound (CRB) of the frequency estimates is given by, [4],

$$\sigma_f^2 = \frac{6f_s^2}{(2\pi)^2\rho N(N^2-1)}. \tag{3}$$

For $N \gg 1$, the asymptotic CRB (ACRB) becomes

$$\sigma_f^2 \approx \frac{6f_s^2}{(2\pi)^2\rho N^3}.$$

The numerical maximization of equation (2) is not a computationally simple task and may suffer from convergence and resolution problems, [1]. Therefore, it is common to estimate the frequency of a sinusoid by a two step process comprising a coarse estimator followed by a fine search, [4]–[10]. The coarse estimation stage is usually implemented using the maximum bin search (MBS) as a coarse approximation of the periodogram maximizer, [11]. This consists of calculating the N-point FFT of the sampled signal and then locating the index of the bin with the highest magnitude.

Various fine frequency estimators have been proposed in the literature. Zakharov and Tozer, [7], present a simple algorithm that consists of an iterative binary search for the true signal frequency. However, they found it necessary to pad the data with zeros to a length of $1.5N$ in order to approach the CRB. Furthermore, the required number of iterations depends on the resolution as well as the operating signal to noise ratio and can be quite large. Quinn, in [1], [5] and [6], proposes a number of estimators that interpolate the true signal frequency using the two discrete Fourier transform (DFT) coefficients either side of the maximum bin. These algorithms, however, have a frequency dependent performance that is worst for a signal frequency coinciding with a bin center. This results in a degradation in performance when they are implemented iteratively, [10], chapter 5.

In this paper we present two new frequency estimators that belong to a family of interpolators amenable to iterative implementation, [10], chapter 6. The first algorithm, denoted Alg1, employs an error functional independently suggested in [10], pp. 129, and [12]. It uses two complex DFT coefficients calculated midway between the standard DFT coefficients. The second algorithm, Alg2, was suggested in [10], pp. 137, and works on the magnitudes of the DFT coefficients. We analyze the new algorithms and show that they have an asymptotic variance that is only 1.0147 times the ACRB. The theoretical results are then verified by simulation.

The paper is organized as follows: In section II we present the details of the frequency estimators including the motivation behind them. In section III we proceed to analyze

TABLE I
ITERATIVE FREQUENCY ESTIMATION BY INTERPOLATION ON FOURIER COEFFICIENTS ALGORITHM

Let	$S = FFT(s)$ and $Y(n) = \|S(n)\|^2,\ n = 0 \dots N-1$
Find	$\hat{m} = \arg\max_n \{Y(n)\}$
Set	$\hat{\delta}_0 = 0$
Loop:	for each i from 1 to Q do $X_p = \sum_{k=0}^{N-1} s(k) e^{-j2\pi k \frac{\hat{m}+\hat{\delta}_{i-1}+p}{N}}, \quad p = \pm 0.5$ $\hat{\delta}_i = \hat{\delta}_{i-1} + h(\hat{\delta}_{i-1})$ where $h(\hat{\delta}_{i-1}) = \frac{1}{2}\mathrm{Re}\left\{\frac{X_{0.5}+X_{-0.5}}{X_{0.5}-X_{-0.5}}\right\}$, for Alg1 or $h(\hat{\delta}_{i-1}) = \frac{1}{2}\frac{\|X_{0.5}\|-\|X_{-0.5}\|}{\|X_{0.5}\|+\|X_{-0.5}\|}$, for Alg2
Finally	$\hat{f} = \frac{\hat{m}+\hat{\delta}_Q}{L} f_s$

the algorithms and derive their asymptotic performances. The convergence properties are also discussed. Section IV shows the simulation results while section V gives the concluding remarks.

II. The Iterative Frequency Estimator

The algorithms are summarized in table I. The coarse search returns the index, $\hat{m}_N$, of the bin with the largest magnitude. Two DFT coefficients at the bin edges are then calculated and used to interpolate the true frequency. The motivation behind each algorithm is easily seen by examining the noiseless case.

Assuming that $\hat{m}_N$ is the index of the true maximum, i.e. $\hat{m}_N = m_N$, the frequency of the signal can be written as

$$f = \frac{\hat{m}_N + \delta_N}{N} f_s \tag{4}$$

where δ_N is a residual in the interval $[-0.5, 0.5]$. The subscript N indicates the dependence of the various parameters on N. In the rest of the paper, unless the dependence on N needs to be emphasized, we drop the subscript for the sake of simplicity of the notation. The goal of the estimator, then, is to obtain an estimate of δ, say $\hat{\delta}$. Consider the DFT coefficients,

$$X_p = \sum_{k=0}^{N-1} s(k) e^{-j2\pi k \frac{\hat{m}+p}{N}}, \quad p = \pm 0.5. \tag{5}$$

Substituting the expression of the sinusoidal signal into (5) and carrying out the necessary manipulations we obtain

$$X_p = e^{j\theta} \frac{1 + e^{j2\pi\delta}}{1 - e^{j2\pi\frac{\delta - p}{N}}} + W_p, \tag{6}$$

where the terms W_p are the Fourier coefficients of the noise. Now for $(\delta - p) \ll N$, equation (6) becomes

$$X_p = b\frac{\delta}{\delta - p} + W_p \tag{7}$$

with b given by

$$b = -Ne^{j\theta}\frac{1 + e^{j2\pi\delta}}{j2\pi\delta}.$$

At this point we ignore the noise terms and proceed to examine the interpolation function of the proposed algorithms. Denote the ratio in the expression of $h(\delta)$ in Alg1 by β. Substituting the expressions for X_p into β and simplifying yields,

$$\begin{aligned}\beta =& \frac{b\frac{\delta}{\delta-0.5} + b\frac{\delta}{\delta+0.5}}{b\frac{\delta}{\delta-0.5} - b\frac{\delta}{\delta+0.5}}\\ =& 2\delta.\end{aligned}$$

Hence, $\hat{\delta} = \beta/2$ can be used as an estimator for the residual frequency δ. However, as we will see in section III-A, it is necessary to take the real value of β in order to obtain a real valued estimate of δ. In a similar way, the motivation behind Alg2 is established as follows; the magnitude of X_p is

$$|X_p| = |b\delta|\left|\frac{1}{\delta - p}\right|.$$

Since $|\delta| \leq 0.5$, the error mapping for Alg2 becomes

$$\frac{1}{2}\frac{|b\delta|\frac{1}{0.5-\delta} - |b\delta|\frac{1}{0.5+\delta}}{|b\delta|\frac{\delta}{0.5-\delta} + |b\delta|\frac{\delta}{0.5+\delta}} = \delta.$$

Again, we find that $\hat{\delta} = h(\delta)$ can be used as an estimator for δ. Note that the bias resulting from the approximation used in going from (6) to (7) is of order N^{-2}. In the following section, we examine the noise performance of the estimators. We show that they are asymptotically unbiased and normally distributed.

III. Theoretical Analysis

A. Asymptotic Performance

The motivation behind each estimator was established in the previous section by examining the noiseless case. We will, now, include the noise terms and derive the asymptotic properties of the estimates. We adopt an analysis strategy similar to that used in [1] and show that both algorithms are asymptotically unbiased and normally distributed.

In the case that the noise is assumed Gaussian, the Fourier coefficients of the noise terms are independent zero mean Gaussian with variance $N\sigma^2$. However, it was shown in [13] and [14] that, given the relaxed assumptions mentioned in the introduction, the noise Fourier coefficients converge in distribution and

$$\limsup_{N\to\infty}\sup_{\lambda}\frac{|W(\lambda)|^2}{N\ln N} \leq 1, \text{almost surely.}$$

Thus, the terms W_p are $O\left(\sqrt{N\ln(N)}\right)$ (for the order notation refer to [15], pp. 421-428).

Now, we have that as $N \to \infty$, $|\hat{m}_N - m_N| \leq 1$ almost surely (a.s.), [1]. In fact, we can show that for $|\delta| < 0.5$, $P\{\hat{m}_N = m_N\} \to 1$ as $N \to \infty$. If $\delta = 0.5$, $P\{\hat{m}_N = m\} = P\{\hat{m}_N = m+1\} = 0.5$ almost surely and either bin is an acceptable choice. The same argument applies for $\delta = -0.5$. Thus, as $N \to \infty$,

$$\delta_N = \hat{m}_N - \frac{Nf}{f_s} \in [-0.5, 0.5] \quad \text{a.s.}$$

Turning our attention to Alg1 and substituting the expression for X_p, shown in equation (7), into β yields, after some simplifications,

$$\beta = \frac{2\delta + \frac{\delta^2 - 0.25}{b\delta}\left(W_{0.5} + W_{-0.5}\right)}{1 + \frac{\delta^2 - 0.25}{b\delta}\left(W_{0.5} - W_{-0.5}\right)}. \tag{8}$$

Since W_p are $O\left(\sqrt{N \ln(N)}\right)$ whereas b is $O(N)$, the term involving δ in the denominator of (8) is of order $O\left(N^{-\frac{1}{2}}\sqrt{\ln(N)}\right)$. Hence, for large N

$$\begin{aligned}\beta =& \left[2\delta + \frac{\delta^2 - 0.25}{b\delta}\left(W_{0.5} + W_{-0.5}\right)\right] \\ & \times \left[1 - \frac{\delta^2 - 0.25}{b\delta}\left(W_{0.5} - W_{-0.5}\right) + O\left(N^{-1}\ln N\right)\right].\end{aligned} \tag{9}$$

Expanding and simplifying, yields

$$\begin{aligned}\beta = 2\delta & \\ & + \frac{\delta^2 - 0.25}{\delta}\text{Re}\left\{\frac{(1-2\delta)W_{0.5} + (1+2\delta)W_{-0.5}}{b}\right\} \\ & + j\frac{\delta^2 - 0.25}{\delta}\text{Im}\left\{\frac{(1-2\delta)W_{0.5} + (1+2\delta)W_{-0.5}}{b}\right\} \\ & + O\left(N^{-1}\ln N\right)\end{aligned} \tag{10}$$

where Re$\{\bullet\}$ and Im$\{\bullet\}$ are respectively the real and imaginary parts of $\bullet$. We clearly see that the real part of β is a noisy estimate of δ. This clarifies the use of the real part of β as an estimator for δ. Thus we set $\hat{\delta} = \frac{1}{2}\text{Re}\{\beta\}$. In fact, taking the real part asymptotically improves the estimation variance by 3dB. Equation (10) implies that the distribution of $\hat{\delta}$ asymptotically follows that of the noise coefficients W_p. Hence, $\hat{\delta}$ is asymptotically unbiased and normally distributed. The asymptotic variance of the estimator is given by

$$\begin{aligned}\text{var}[\hat{\delta}] =& \frac{1}{4}\frac{\left(\delta^2 - 0.25\right)^2}{|b|^2\delta^2}\left\{(1-2\delta)^2\text{var}[\text{Re}\left\{W_{0.5}\right\}]\right. \\ & \left. + (1+2\delta)^2\text{var}[\text{Re}\left\{W_{-0.5}\right\}]\right\} \\ =& \frac{1}{4}\frac{\sigma^2}{N}\frac{\pi^2\left(\delta^2 - 0.25\right)^2\left(4\delta^2+1\right)}{\cos^2(\pi\delta)},\end{aligned} \tag{11}$$

where the second equality follows from the fact that, under the Gaussianity assumption, var[Re $\{W_{0.5}\}$] $=$ var[Re $\{W_{-0.5}\}$] $= N\sigma^2/2$, and

$$|b|^2 = N^2\frac{\cos^2(\pi\delta)}{(\pi\delta)^2} \tag{12}$$

The performance of Alg2 can be obtained in a similar fashion. Let $Y_p = |X_p|$. Thus,

$$Y_p = \left|b\frac{\delta}{\delta - p}\right|\left|1 + \frac{\delta - p}{b\delta}W_p\right|. \tag{13}$$

The second factor in the above expression can be expanded as follows

$$\left|1 + \frac{\delta - p}{b\delta}W_p\right| = \sqrt{1 + \frac{(\delta-p)^2}{|b|^2\delta^2}|W_p|^2 - 2\frac{\delta - p}{\delta}\text{Re}\left\{\frac{W_p}{b}\right\}} \tag{14}$$

Upon examination of the two terms under the square root, we find that their relative orders change as $|\delta| \to 0.5$. This leads us to divide the interval $[-0.5, 0.5]$ into two regions, Δ_1 and Δ_2, defined for some $a > 0$ and $\nu > 0$, as shown

$$\Delta_1 = \left\{\delta; |\delta| \le 0.5 - aN^{-\nu}\right\} \tag{15}$$

and

$$\Delta_2 = \left\{\delta; 0.5 - aN^{-\nu} \le |\delta| \le 0.5\right\}. \tag{16}$$

For $\delta \in \Delta_1$, Re$\{W_p/b\}$ is of order $O(N^{-\frac{1}{2}}\sqrt{\ln N})$, whereas $|W_p/b|^2$ is $O(N^{-1}\ln N)$. Therefore, ignoring the lower order term involving $|W_p|^2$ and using the fact that for $x \ll 1$, $\sqrt{1+x} = 1 + x/2 + O(x^2)$, we obtain

$$Y_p = \left|b\frac{\delta}{\delta - p}\right|\left[1 - \frac{\delta - p}{\delta}\text{Re}\left\{\frac{W_p}{b}\right\}\right] + o(1). \tag{17}$$

Substituting Y_p into the error mapping for Alg2 and carrying out the analysis in a similar way as was done for Alg1, we find that

$$\begin{aligned}\hat{\delta} =& \delta + \frac{1}{2}\frac{\delta^2 - 0.25}{\delta}\left[(2\delta - 1)\text{Re}\left\{\frac{W_{0.5}}{b}\right\}\right. \\ & \left. + (2\delta + 1)\text{Re}\left\{\frac{W_{-0.5}}{b}\right\}\right].\end{aligned} \tag{18}$$

This result is similar to the estimator expression of Alg1 obtained by taking half the real part of (10). In fact for $\delta \in \Delta_1$ the performances of the two algorithms are statistically equivalent since b is a complex constant and does not affect the statistics of the noise coefficients W_p. Now, turning our attention to region Δ_2, we find that the estimator is biased. We consider here the case where $\delta \to 0.5$, the other case is similar. As $\delta \to 0.5$, the orders of the terms in the expression of $Y_{0.5}$ are preserved. However, looking at $Y_{-0.5}$ we find that there is a value of δ close to 0.5 after which the term in $|W_{-0.5}|^2$ starts to dominate that in Re$\{W_{-0.5}\}$. The estimator then becomes biased since E $\left[|W_{-0.5}|^2\right] \ne 0$. We take this value of δ to be the boundary between regions Δ_1 and Δ_2. Let $\zeta = 0.5 - \delta$. Now the term involving $|W_{-0.5}|^2$ is of order $O(\zeta^{-2}N^{-1}\ln N)$, whereas that in Re$\{W_{-0.5}\}$ is $O(\zeta^{-1}N^{-\frac{1}{2}}\sqrt{\ln N})$. As a definition, we take a quantity Q_2 to dominate another quantity Q_1 if $Q_1/Q_2 = o(1)$. A function that satisfies this requirement is $\phi = 1/\sqrt{\ln N}$. Note that this choice of ϕ is arbitrary and any other function that is $o(1)$ could have been used. Using this definition, we find that the term in $|W_{-0.5}|^2$ dominates that in Re$\{W_{-0.5}\}$ when $\zeta = N^{-\frac{1}{2}}$. Thus, the boundary between Δ_1 and Δ_2 is given by $0.5 - N^{-\frac{1}{2}}$. The resulting bias of the estimator for $\delta \in \Delta_2$ is $O(\ln N)$. On the other hand, the width of region Δ_2 is $o(N^{-\frac{1}{2}})$. As Δ_2 vanishes faster than the growth rate of the bias, the asymptotic result of region Δ_1 holds and the algorithm is asymptotically unbiased with a performance that is identical to that of Alg1.

Finally, we have the following theorem:

Theorem 1: Let $\hat{\delta}_N$ be given by the error functionals of Alg1 or Alg2 (with $\delta_N \in \Delta_1$ for Alg2) and let $\hat{f}_N$ be defined as

$$\hat{f}_N = \frac{\hat{m}_N + \hat{\delta}_N}{N} f_s$$

then $\sigma^{-1}\left(\hat{f} - f\right)$ is asymptotically standard normal with σ given by

$$\sigma^2 = \frac{f_s^2}{4N^3\rho} \frac{\pi^2\left(\delta^2 - 0.25\right)^2\left(4\delta^2+1\right)}{\cos^2(\pi\delta)}.$$

A useful indicator of the algorithm performance is the ratio of its asymptotic variance to the ACRB. This is,

$$R = \frac{\pi^4}{6} \frac{\left(\delta^2 - 0.25\right)^2\left(4\delta^2+1\right)}{\cos^2(\pi\delta)}. \tag{19}$$

The error functionals are then seen to have identical performances. The ratio of the variance of the estimates, for SNRs above the threshold, is dependent on δ, but independent of the SNR. Furthermore, it has a minimum of 1.0147 for $\delta = 0$.

B. Iterative Implementation

In the previous section, we showed that the performance of the interpolation functions of both estimators depend on the true signal frequency. The iterative procedure of table I reduces this frequency dependence and improves the performance of the algorithm. The estimate of the residual obtained at each iteration is removed from the signal and the estimator reapplied to the compensated data. In this section we show that the estimators are well behaved and the procedure converges in two iterations. This allows for a computationally efficient algorithm with a performance that is only marginally above the CRB.

We will only consider the iterative estimator constructed using Alg1. A similar argument can be constructed for Alg2, [10], pp. 194-199. Let the true value of the residual be denoted by δ_0. Now $h(\delta)$ can be written as

$$h(\delta) = \frac{\sin\left(\frac{2\pi}{N}(\delta_0 - \delta)\right)}{2\sin\left(\frac{\pi}{N}\right)} \left[1 + O\left(N^{-\frac{1}{2}}\sqrt{\ln N}\right)\right]. \tag{20}$$

Expanding $h(\delta)$ into a Taylor series about δ_0 gives

$$h(\delta) = (\delta - \delta_0)h'(\delta_0)\left[1 + O(N^{-\frac{1}{2}}\sqrt{\ln N})\right] \tag{21}$$

where

$$\begin{aligned} h'(\delta_0) =& -\frac{\pi}{N\sin\left(\frac{\pi}{N}\right)}\left[1 + O\left(N^{-\frac{1}{2}}\sqrt{\ln N}\right)\right] \\ =& -1 + O\left(N^{-\frac{1}{2}}\sqrt{\ln N}\right). \end{aligned} \tag{22}$$

The estimation function $\psi(\delta) = \delta + h(\delta)$ becomes

$$\psi(\delta) = \delta_0 + (\delta - \delta_0)O(N^{-\frac{1}{2}}\sqrt{\ln N}). \tag{23}$$

Now for any $\delta_1, \delta_2 \in [-0.5, 0.5]$, we have

$$\begin{aligned} |\psi(\delta_1) - \psi(\delta_2)| =& |\delta_1 - \delta_2| O(N^{-\frac{1}{2}}\sqrt{\ln N}) \\ \leq& \alpha|\delta_1 - \delta_2| \end{aligned}$$

with $\alpha < 1$. Thus, the iterative procedure constitutes a contractive mapping on [-0.5,0.5]. Also, it has a unique fixed point at δ_0. That is, $\psi(\delta_0) = \delta_0$. Consequently, the fixed point theorem, [16], pp. 133, ensures that the algorithm of table I converges to the fixed point. The residual input to the algorithm at the i^{th} iteration is $\delta_0 - \hat{\delta}_{i-1}$. Let the variance expression of the estimator, shown in equation (11), be denoted by $g(\delta)$. The variance of the estimate, $\hat{\delta}_i$, at the i^{th} iteration is given by $g(\delta_0 - \hat{\delta}_{i-1})$. Thus, the limiting variance of the estimator is

$$\begin{aligned} \mathrm{var}[\hat{\delta}_\infty] =& \lim_{i\to\infty} g(\delta_0 - \hat{\delta}_{i-1}) \\ =& g(0), \end{aligned} \tag{24}$$

where the last result follows from the fact that $\lim_{i\to\infty}\hat{\delta}_i = \delta_0$ and $g(\delta)$ is continuous on $[-0.5, 0.5]$. Now we turn our attention to the stopping criteria. The CRB for δ, which is $O(N^{-\frac{1}{2}})$, forms a lower bound on the estimation variance and no further gain is achievable once the residual frequency is of order lower than it. Therefore, it is reasonable to stop the estimator once the residual is $o(N^{-\frac{1}{2}})$. Let this iteration number be Q. Starting with an initial estimate, $\hat{\delta}_0 = 0$, and using equation (23), the estimate after the first iteration, $\hat{\delta}_1$ is given by,

$$\hat{\delta}_1 = \delta_0\left[1 + O(N^{-\frac{1}{2}}\sqrt{\ln N})\right]$$

and the residual is $\hat{\delta}_1 - \delta_0 = O(N^{-\frac{1}{2}}\sqrt{\ln N})$. This is still of order higher than the CRB. Looking at the estimate after the second iteration, we have

$$\hat{\delta}_2 = \delta_0\left[1 + O(N^{-1}\ln N)\right] \tag{25}$$

and the residual is $\hat{\delta}_2 - \delta_0 = O(N^{-1}\ln N)$, which is now $o(N^{-\frac{1}{2}})$. Thus, only two iterations are needed for the residual to become of lower order than the CRB. We say that the algorithm has converged after two iterations. These results are summarized by the following theorem.

Theorem 2: The iterative procedure defined using Alg1 or Alg2, as shown in table I, converges with the following properties

- The fixed point of convergence is the true offset, δ_0
- The procedure takes 2 iterations for the residual error to become $o(N^{-\frac{1}{2}})$
  and
- The limiting ratio of the variance of the estimator to the asymptotic CRB is $\pi^4/96$ or 1.0147 uniformly over the interval [-0.5,0.5].

At this stage, we note that, as mentioned in the introduction, the results derived in this paper are valid under the more general and relaxed noise assumptions stated by Quinn in [1].

IV. Simulation Results

The algorithms presented above were implemented and simulated. The number of samples used in the simulation was $N = 1024$. Fig. 1 displays the theoretical and simulation results on the ratio of the variance of the estimates to the ACRB versus δ for one and two iterations. The signal to noise ratio for this simulation was set to 0dB. We see that for

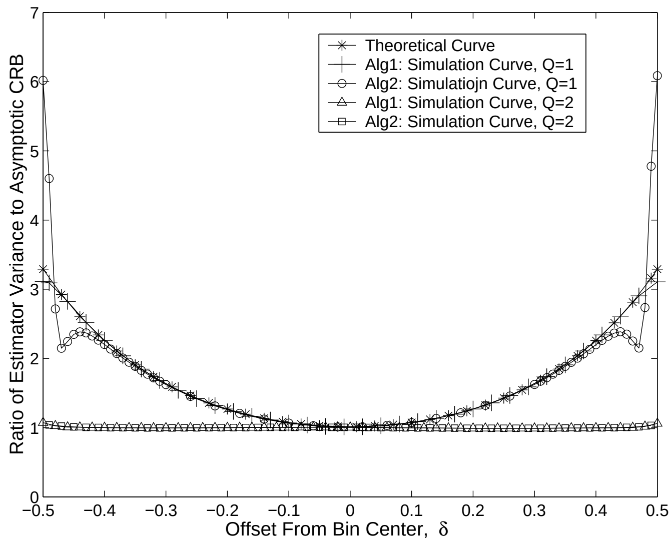

Fig. 1. Plot of the ratio of the variance of the Alg1 and Alg2 to the asymptotic CRB as a function of δ, the frequency offset from the bin center. Simulation curves for 1 and 2 iterations are shown. The theoretical curve is also displayed. 10000 runs were averaged at a signal to noise ratio of 0dB.

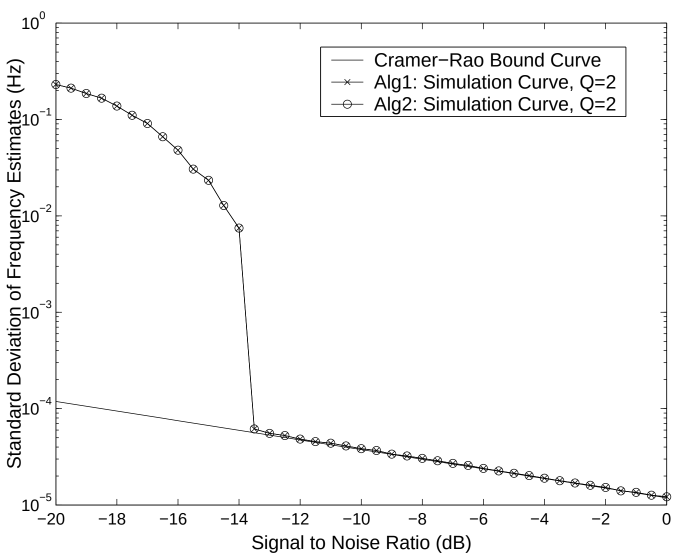

Fig. 2. Plot of the standard deviation of the frequency estimation error for Alg1 and Alg2 as a function of the signal to noise ratio. The Cramer Rao Bound curve is also shown. 10000 runs at each signal to noise ratio were averaged.

$Q = 1$, the simulation and theoretical results agree closely. For Alg2, the boundary between regions Δ_1 and Δ_2 is clearly visible. The plot also shows that after the second iteration, the performance of both algorithms is uniform over the entire interval. The ratio of the variance of both estimators is, as expected, very close to the theoretical value of 1.0147. Fig. 2 presents the simulation results of the noise performance of both algorithms as a function of the signal to noise ratio. The CRB curve is shown for the purpose of comparison. Both algorithms exhibit almost identical performances that are on the CRB curve. The threshold effect that is characteristic of the ML estimator and which results from the coarse estimation stage, is visible.

V. Conclusion

In this paper we have proposed and analyzed two new estimators for the frequency of a complex exponential in additive noise. The estimators consist of a coarse search followed by a fine search algorithm. The coarse search is implemented using the standard Maximum Bin Search. Two new fine search algorithms have been proposed and their asymptotic performances derived. The estimator were implemented iteratively and the resulting procedure shown to converge to the true signal frequency. The estimation variance of the iterative algorithm was also shown to converge asymptotically to its minimum value. This results in an improvement in the performances of the estimators when implemented iteratively. The number of iterations required for convergence was found to be 2 for both algorithms. Hence the iterative estimator has a computational load of the same order as the FFT. Finally, the theoretical results were verified by simulations.

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