Adaptive Array Thinning for Enhanced DOA Estimation

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***Abstract*—Antenna array configurations play an important role in direction of arrival (DOA) estimation. In this letter, performance enhancement of DOA estimation is achieved by reconfiguring the multi-antenna receiver through an antenna selection strategy. We derive the Cramer-Rao Bound (CRB) in terms of the selected antennas and associated subarray for both peak sidelobe level (PSL) constrained isotropic and directional arrays in single source cases. Since directional arrays are angle dependent, a Dinklebach type algorithm and convex relaxation are introduced to maintain the optimum selection by adaptively reconfiguring the directional subarrays using semi-definite programming. Simulation results validate the effectiveness of the proposed antenna selection strategy.**

***Index Terms*—Cramer-Rao bound, antenna selection, directional array, isotropic array, Dinklebach algorithm**

I. Introduction

ESTIMATING the direction of arrival (DOA) using antenna arrays has been an important topic in signal processing with diverse applications. DOA estimation accuracy is dependent not only on the employed algorithm, but also on the receiver array configuration. Extensive research has been devoted to investigate the effect of array configuration on DOA estimation performance for both near-field [1], [2] and far-field scenarios [3]. The Cramer-Rao Bound (CRB) is commonly used as a metric for characterising the estimation performance in terms of the array configuration [4]. A compact formula of the CRB in terms of antenna positions for isotropic 2D and 3D arrays was derived in [5]–[8], and a Bayesian CRB approach for a single source with known prior probability distribution was proposed in [9]. A study of the CRB for 2D arrays, presented in [10]–[12], showed that the optimum array is V-shaped under the assumptions of equal inter-element spacing and concave array geometry. The design of optimum directional arrays was also introduced in [11], but the work considered only the most favourable direction and the proposed exhaustive search strategy places limitations on the practicality of this method.

Beside the computational cost, the prohibitive hardware cost of large arrays, where a separate receiver is used for each antenna, is impractical and presents a significant limiting factor. Therefore, we maximize the DOA estimation performance for a given reduced number of antennas (i.e. reduced hardware and computational cost) by varying the array geometry [3]. In order to realize the array reconfigurability, we thin a full array by acting on a sequence of Radio Frequency (RF) switches. The problem of sensor selection for a desired CRB with the smallest number of antennas was considered in [13], albeit for a uniform linear array. We, on the other hand, generalize in this work the antenna selection to achieve the lowest CRB with a fixed number of antennas over arbitrarily shaped arrays that are either isotropic or directional. Since the non-uniformity of selected antennas typically results in high sidelbobes, the trade-off between peak sidelobe suppression and estimation accuracy is controlled though the Spatial Correlation Coefficient (SCC) [14]. Continuously changing operational environments in radar, satellite communication etc. require adaptive enhanced interference localization for subsequent cancellation. It is well-known that antenna selection is essentially an NP-hard combinatorial optimization. Optimum peak sidelobe level (PSL) constrained isotropic subarrays may be found using an exhaustive search, whereas directional subarrays are angle dependent and require a polynomial-time selection algorithm to implement array thinning adaptively. Therefore, we employ an effective Dinklebach-type algorithm and convex relaxation for antenna selection through semi-definite programming.

The remainder of this letter is organized as follows: The mathematical model is derived in Section II. In sections III and IV, the selection of optimum PSL constrained isotropic and directional subarrays are introduced respectively. Simulation results are presented in section V. The last section gives some concluding remarks.

II. Mathematical Model

Consider a set of N antennas located in the (x, y) plane. We associate each antenna with the x and y coordinates x_n and y_n, $n = 1, \cdots, N$ respectively. A single narrow-band signal $s(t)$ with wavelength λ is impinging on the array from azimuth $\phi \in [0, 2\pi]$ and elevation angle $\theta \in [0, \pi/2]$. The steering vector of the signal is,

$$\mathbf{a} = [e^{jk_0(x_1 u_x + y_1 u_y)}, ..., e^{jk_0(x_N u_x + y_N u_y)}], \tag{1}$$

where $k_0 = 2\pi/\lambda$, $u_x = \sin\theta\cos\phi$ and $u_y = \sin\theta\sin\phi$. Assuming omni-directional antennas and far-field sources, the received signal can then be expressed as

$$\mathbf{x}(t) = \mathbf{a}s(t) + \mathbf{n}(t),\ t = 1, ..., T. \tag{2}$$

The model is referred to as deterministic if $s(t)$ is a deterministic unknown signal, and random if $s(t)$ is assumed random. The choice of either model depends on the application. Since the CRBs of both data models have the same dependence on the array structure [5], we consider the random waveform model

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in what follows. We assume both the estimated signal and noise to be Gaussian with zero mean, and constant variances σ_s^2 and σ_n^2, respectively. The signal-to-noise ratio (SNR) is defined as $\rho = \sigma_s^2/\sigma_n^2$.

Given the full array, we define the following vectors using the antenna positions,

$$\mathbf{x} = [x_1, \cdots, x_N]^T, \mathbf{y} = [y_1, \cdots, y_N]^T, \mathbf{x}_x = [x_1^2, \cdots, x_N^2]^T, \quad (3)$$

$$\mathbf{y}_y = [y_1^2, \cdots, y_N^2]^T, \mathbf{x}_y = [x_1 y_1, \cdots, x_N y_N]^T. \quad (4)$$

Let the Fisher Information Matrix (FIM) for the estimation of elevation angle θ and azimuth angle ϕ be $\mathbf{J}$, i.e.,

$$\mathbf{J} = \begin{bmatrix} J_{\theta\theta} & J_{\theta\phi} \\ J_{\phi\theta} & J_{\phi\phi} \end{bmatrix}. \quad (5)$$

We define $\mathbf{w} \in \{0, 1\}^N$ to be a binary selection vector where an entry of zero means that the corresponding antenna is discarded and one means it is selected. Suppose the number of active (selected) antennas is K, the center of gravity of the thinned array is defined as [7],

$$x_c = \frac{1}{K}\sum_{i=1}^{N} \mathbf{w}(i)x_i = \frac{1}{K}\mathbf{w}^T\mathbf{x}, \; y_c = \frac{1}{K}\sum_{i=1}^{N} \mathbf{w}(i)y_i = \frac{1}{K}\mathbf{w}^T\mathbf{y}. \quad (6)$$

We assume, without loss of generality, that the centre of the coordinate system is colocated with the centre of gravity of the thinned array [7], i.e.,

$$x_c = \frac{1}{K}\mathbf{w}^T\mathbf{x} = 0, \quad y_c = \frac{1}{K}\mathbf{w}^T\mathbf{y} = 0. \quad (7)$$

As shown in [6], [7], [15], the CRB is a function of the array configuration through the following parameters involving the selected antenna positions,

$$Q_{xx} = \mathbf{w}^T\mathbf{x}_x, Q_{yy} = \mathbf{w}^T\mathbf{y}_y, Q_{xy} = \mathbf{w}^T\mathbf{x}_y. \quad (8)$$

The FIM, $\mathbf{J}$, can then be expressed in terms of the selected antennas as follows [6], [7], [15],

$$J_{\theta\theta} = \mathcal{G}\cos^2\theta\left\{\cos^2\phi Q_{xx} + \sin^2\phi Q_{yy} + \sin 2\phi Q_{xy}\right\}, \quad (9)$$

$$J_{\phi\phi} = \mathcal{G}\sin^2\theta\left\{\sin^2\phi Q_{xx} + \cos^2\phi Q_{yy} - \sin 2\phi Q_{xy}\right\}, \quad (10)$$

and

$$J_{\phi\theta} = \frac{\mathcal{G}}{4}\sin 2\theta\left\{\sin 2\phi(Q_{xx} - Q_{yy}) - 2\cos 2\phi Q_{xy}\right\}, \quad (11)$$

where $J_{\theta\phi} = J_{\phi\theta}$ and $\mathcal{G} = 2N\rho^2k_0^2/(1+\rho N)$ is angle independent.

III. Optimum PSL Constrained Isotropic Subarray

The single source CRB of an isotropic array is independent of the azimuth angle in [8]. The configuration of an optimum isotropic array is independent on arrival directions, i.e. both elevation and azimuth angles. Then isotropic arrays are obtained when, [7],

$$Q_{xy} = 0, \quad Q_{xx} = Q_{yy} = Q. \quad (12)$$

Combining the condition in Eq. (12) with Eqs. (9-11) implies that the FIM is a diagonal matrix and the CRB becomes,

$$\mathbf{C} = \mathbf{J}^{-1} = \frac{1}{\mathcal{G}}\begin{bmatrix} 1/(\cos^2\theta Q) & 0 \\ 0 & 1/(\sin^2\theta Q) \end{bmatrix}. \quad (13)$$

It should be noted that an isotropic subarray satisfying the symmetric condition of Eq. (12) does not always exist.

Now proceeding from Eq. (13), and minimizing the trace of the CRB for optimum array thinning [16], [17], we have that,

$$\min\left\{\mathrm{tr}(\mathbf{C}) = \frac{1}{\sin^2\theta\cos^2\theta Q}\right\} \Leftrightarrow \max\{Q\}, \quad (14)$$

where tr(•) denotes the trace of the matrix •. It is clear from Eq. (14) that the optimum isotropic thinned array includes the boundary antennas, which can guarantee the largest aperture. This observation agrees with the conclusion in [18] for linear arrays. But, the optimum isotropic subarrays that consist of the boundary antennas typically exhibit high sidelobes [19]–[21]. In order to solve this problem, we utilise the spatial correlation coefficients (SCC) [14], [22] to control the trade-off between the estimation variance and the synthesized beampattern. Since the SCC denotes the cross correlation between the steering vectors of two separated incoming sources, it is only dependent on electrical angle differences, i.e. $\Delta\mathbf{u} = [\Delta u_x, \Delta u_y]^T$. Let $\Delta\mathbf{u}_{i,j} = [\Delta u_x^i, \Delta u_y^j] \in [-2, 2] \times [-2, 2], i, j = 1, ..., L_1(L_2)$ be the samples of angle differences in u-space. Then the correlation steering vector $\mathbf{v}_{i,j}$ is defined as, [14],

$$\mathbf{v}_{i,j} = e^{jk_0(\Delta u_x^i \mathbf{x} + \Delta u_x^j \mathbf{y})}, i, j = 1, ..., L_1(L_2). \quad (15)$$

The samples, $\Delta\mathbf{u}_{i,j}$, can be set to be the specified electrical angular region with the constrained PSL.

Now, we consider a set of subarrays with K antennas and the array center of gravity colocated with the center of the coordinate system,

$$\mathcal{S} = \{\mathbf{w} \in \{0, 1\}^N : \mathbf{w}^T\mathbf{x} = 0; \mathbf{w}^T\mathbf{y} = 0; \mathbf{1}^T\mathbf{w} = K\}, \quad (16)$$

where $\mathbf{1}$ is a vector with all ones. Note that the set $\mathcal{S}$ comprises the extreme points of the polyhedra,

$$\mathcal{P} = \{\mathbf{w} \in [0, 1]^N : \mathbf{w}^T\mathbf{x} = 0; \mathbf{w}^T\mathbf{y} = 0; \mathbf{1}^T\mathbf{w} = K\}. \quad (17)$$

The problem of determining the optimum isotropic subarray with constrained PSL is formulated as,

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^T\mathbf{x}_x \\ \text{s.t.} \quad & \mathbf{w} \in \mathcal{S}; \quad \mathbf{w}^T\mathbf{x}_y = 0; \quad \mathbf{w}^T(\mathbf{x}_x - \mathbf{y}_y) = 0; \\ & \mathbf{w}^T\mathbf{V}_{i,j}\mathbf{w} \le \delta_{i,j}, i, j = 1, ..., L_1(L_2), \end{aligned} \quad (18)$$

where $\mathbf{V}_{i,j} = \mathrm{real}(\mathbf{v}_{i,j}\mathbf{v}_{i,j}^H)$ and $\delta_{i,j} < 1$ is the desired normalised sidelobe power level with respect to the mainlobe. The problem in Eq. (18) is convex programming, except for the binary constraints $\mathbf{w} \in \{0, 1\}^N$. Since the isotropic array is independent of the estimated angle, the optimum solution of Eq. (18) may be calculated off-line through an exhaustive search. In order to reduce computational load, another method of solving Eq. (18) is to relax the binary constraints through the difference of two convex sets (DCS), which is a polynomial-time algorithm with the detailed implementation procedure given in [14]. Here, we formulate the DCS for antenna selection in the $(k+1)^{th}$ iteration as follows,

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^T(\mathbf{x}_x + 2\mu\mathbf{w}_k - \mu\mathbf{1}) \\ \text{s.t.} \quad & \mathbf{w} \in \mathcal{P}; \quad \mathbf{w}^T\mathbf{x}_y = 0; \quad \mathbf{w}^T(\mathbf{x}_x - \mathbf{y}_y) = 0; \\ & \mathbf{w}^T\mathbf{V}_{i,j}\mathbf{w} \le \delta_{i,j}, i, j = 1, ..., L_1(L_2), \end{aligned} \quad (19)$$

where μ is a trade-off parameter which compromises between the solution sparseness and the CRB.

IV. Optimum Directional Subarray

As shown in Eqs. (9)-(11), the FIM depends on the array geometry and the source DOAs. Therefore, the optimum array for one angle is not necessarily optimum for other angles. If prior information on the source DOAs is available, it would be desirable to select directional subarrays that optimize the estimation performance in a certain neighbourhood of the privileged direction, including both elevation and azimuth. For example, some prior information on the DOAs can be obtained utilising PSL constrained isotropic subarrays. Since the estimated angle is narrowed down within a small range, it becomes unnecessary to consider peak sidelobe suppression for directional subarrays.

A. Problem Formulation

Assume the neighbourhood of interest is around the angle $[\theta, \phi]$. After mathematical manipulations, the CRB of θ is

$$C_{\theta\theta} = \frac{1}{\mathcal{G}} \cdot \frac{1}{\cos^2\theta} \cdot \frac{\sin^2\phi Q_{xx} + \cos^2\phi Q_{yy} - \sin 2\phi Q_{xy}}{Q_{xx}Q_{yy} - Q_{xy}^2}, \quad (20)$$

similarly, the CRB of ϕ is

$$C_{\phi\phi} = \frac{1}{\mathcal{G}} \cdot \frac{1}{\sin^2\theta} \cdot \frac{\cos^2\phi Q_{xx} + \sin^2\phi Q_{yy} + \sin 2\phi Q_{xy}}{Q_{xx}Q_{yy} - Q_{xy}^2}. \quad (21)$$

Unlike [15], where the volume of the confidence region was introduced for azimuth-invariant cases and was defined only in terms of the determinant of the FIM, i.e., the denominator $Q_{xx}Q_{yy} - Q_{xy}^2$, we utilise the trace of the CRB as a metric,

$$\mathrm{tr}(\mathbf{C}) = \frac{1}{\mathcal{G}} \cdot \frac{1}{Q_{xx}Q_{yy} - Q_{xy}^2} \cdot \left[\alpha Q_{xx} + \beta Q_{yy} + \zeta Q_{xy}\right], \quad (22)$$

where

$$\alpha = \frac{\sin^2\phi}{\cos^2\theta} + \frac{\cos^2\phi}{\sin^2\theta}, \beta = \frac{\cos^2\phi}{\cos^2\theta} + \frac{\sin^2\phi}{\sin^2\theta}, \zeta = \frac{\sin 2\phi}{\sin^2\theta} - \frac{\sin 2\phi}{\cos^2\theta}.$$

The optimization problem can be formulated as,

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{\mathbf{w}^T(\tilde{\alpha}\mathbf{x}_x\mathbf{1}^T + \tilde{\beta}\mathbf{y}_y\mathbf{1}^T + \tilde{\zeta}\mathbf{x}_y\mathbf{1}^T)\mathbf{w}}{\mathbf{w}^T(\mathbf{x}_x\mathbf{y}_y^T - \mathbf{x}_y\mathbf{x}_y^T)\mathbf{w}} \\ \text{s.t.} \quad & \mathbf{w} \in \mathcal{S}; \end{aligned} \quad (23)$$

where $\tilde{\alpha} = \alpha/K$, $\tilde{\beta} = \beta/K$ and $\tilde{\zeta} = \zeta/K$. Contrary to isotropic subarrays, an optimum K-antenna directional subarray always exists. The objective function in Eq. (23) is a quadratic fractional, which makes Eq. (23) difficult to solve. In order to alleviate the problem, we introduce another variable $\mathbf{W}$ along with the rank-one constraint $\mathbf{W} = \mathbf{w}\mathbf{w}^T$. Then the binary constraint can be reformulated as,

$$\mathrm{tr}(\mathbf{W}\mathbf{E}_i) - \mathbf{e}_i^T\mathbf{w} = 0, i = 1, \cdots, N \quad (24)$$

where $\mathbf{e}_i$ is the i^{th} unit vector with the i^{th} entry being one and all others being zero and $\mathbf{E}_i = \mathbf{e}_i\mathbf{e}_i^T$. Correspondingly, the set $\mathcal{S}$ is rewritten as,

$$\begin{aligned} \mathcal{S} \quad = \quad & \{\mathbf{w}, \mathbf{W} : \mathbf{w}^T\mathbf{x} = 0; \mathbf{w}^T\mathbf{y} = 0; \mathbf{1}^T\mathbf{w} = K; \\ & \mathrm{trace}(\mathbf{W}\mathbf{E}_i) - \mathbf{e}_i^T\mathbf{w} = 0, i = 1, \cdots, N\}. \end{aligned}$$

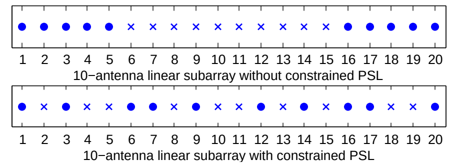

Fig. 1: Optimum 10-antenna linear subarrays.

We rewrite Eq.(23) by relaxing the rank-one constraint as follows,

$$\begin{aligned} \min \quad & \frac{\mathrm{tr}(\mathbf{W}\mathbf{N}_u)}{\mathrm{tr}(\mathbf{W}\mathbf{D}_e)} \\ \text{s.t.} \quad & \{\mathbf{w}, \mathbf{W}\} \in \mathcal{S}, \quad \mathbf{W} \geq \mathbf{w}\mathbf{w}^T, \end{aligned} \quad (25)$$

where $\mathbf{N}_u = \tilde{\alpha}\mathbf{x}_x\mathbf{1}^T + \tilde{\beta}\mathbf{y}_y\mathbf{1}^T + \tilde{\zeta}\mathbf{x}_y\mathbf{1}^T$ and $D_e = \mathbf{x}_x\mathbf{y}_y^T - \mathbf{x}_y\mathbf{x}_y^T$. An exhaustive searching within $\mathcal{S}$ can be conducted for small arrays to find the optimum directional subarray, while for large arrays, we propose a Dinklebach-type algorithm for adaptive directional subarray selection.

B. Dinkelbach-type Algorithm

The Dinkelbach-type algorithm is based on a theorem concerning the relationship between factional and parametric programming [23]. The parametric objective function is transformed from the fraction in Eq. (25),

$$F(\eta) = \mathrm{tr}(\mathbf{W}\mathbf{N}_u) - \eta\,\mathrm{tr}(\mathbf{W}\mathbf{D}_e). \quad (26)$$

The detailed derivation and performance analysis can be found in [23]. Here we give an outline of the procedure as follows.

Step 1: Initialize η_1 and the termination threshold $\varepsilon = 0.01$;

Step 2: Solve the following minimization problem to obtain global solutions $\mathbf{w}_k$, $\mathbf{W}_k$ and the optimum value $F(\eta_k)$:

$$\begin{aligned} \min \quad & F(\eta_k) \\ \text{s.t.} \quad & \{\mathbf{w}, \mathbf{W}\} \in \mathcal{S}; \quad \mathbf{W} \geq \mathbf{w}\mathbf{w}^T. \end{aligned} \quad (27)$$

Step 3: If $F(\eta_k) \leq \varepsilon$, then terminate. Otherwise, let

$$\eta_{k+1} = \frac{\mathrm{tr}(\mathbf{W}_k\mathbf{N}_u)}{\mathrm{tr}(\mathbf{W}_k\mathbf{D}_e)}. \quad (28)$$

and return to Step 2.

The selection vector $\mathbf{w}$ is generated by setting the K largest entries to be one. The initial η_1 can take the corresponding value of the isotropic subarray to accelerate convergence rate.

V. Simulation Results

Firstly, we select a 10-antenna subarray from a 20-antenna uniform linear array (ULA) for DOA estimation. Since the effect of the array geometry on the CRB for a linear array can be separated from the arrival angle, the optimum linear subarray is always isotropic. The two optimum subarrays with and without constrained PSL are shown in Fig. 1. The subarray without constrained PSL comprises two clusters of antennas, one at each end of the linear array. For the subarray with constrained PSL, the squared SCC value is set to be $\delta = 0.5$, which implies the PSL is -6dB as shown in Fig. 2. Finally, the estimation variance versus SNR for the two subarrays is

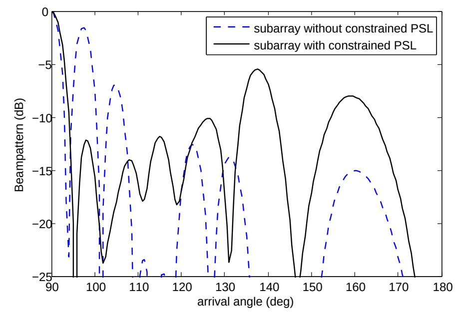

Fig. 2: Beampatterns of two linear subarrays.

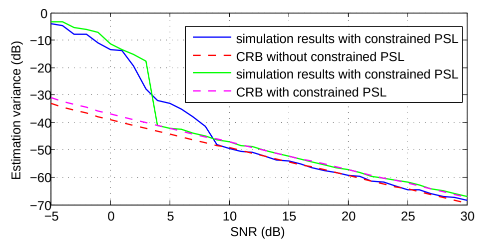

Fig. 3: Estimation variance of two subarrays versus SNR. Each point is averaged by 500 Monte-Carlo runs.

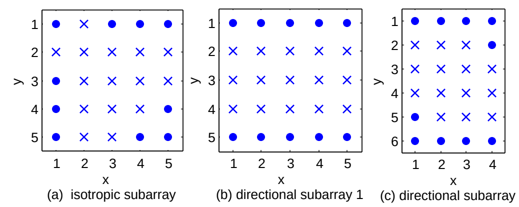

Fig. 4: Three 10-antenna planar subarrays.

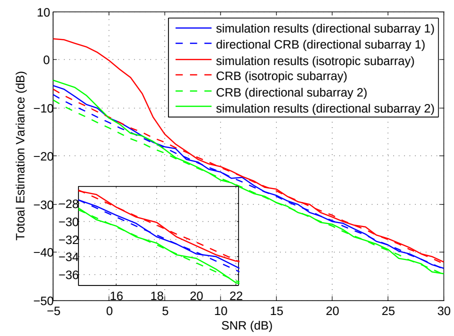

Fig. 5: Total estimation variance for three subarrays versus SNR. Each point is averaged by 500 Monte-Carlo runs.

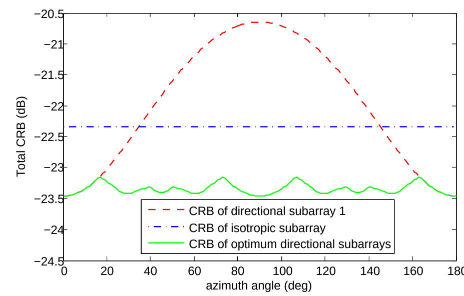

Fig. 6: Total CRB versus the azimuth angle. The SNR is 10dB.

shown in Fig. 3. The subarray with constrained PSL has 2.1 dB performance loss compared to the other subarray, however it exhibits a 5 dB smaller threshold value due to the well synthesized beampattern. The ULA exhibits 3.14 dB and 1.04 dB smaller estimation variance than the two subarrays with and without PSL constraints respectively, although with 10 more antennas.

Next, we select a 10-antenna subarray from a 6×4 rectangular planar array. Due to the symmetric requirement, it is impossible to select an isotropic 10-antenna subarray from this rectangular array. Thus we assume another 5×5 square array for comparison. The desired signal is impinging on the array from an azimuth of $10°$ and elevation of $175°$. The isotropic and two directional subarrays are shown in Fig. 4. Note that the directional subarray configurations may have grating lobes, which does not affect the DOA estimation performance with some prior knowledge of the arrival angle. The total estimation variance, given by the sum of the elevation and azimuth, for the three subarrays are shown in Fig. 5. The first directional subarray has 1.11dB better performance than the isotropic subarray, while the second directional subarray exhibits the best estimation performance with 2.26 dB smaller estimation variance compared to the isotropic one. It should be noted that the low threshold value for both directional subarrays results from the prior information.

Finally, we investigate the relationship between the optimum directional and isotropic subarrays. It is clear from Eq. (22) that when ζ is close to zero and α is close to β, the optimum directional subarray is essentially the isotropic one. This occurs when the estimated elevation angle is in the neighbourhood of $45°$. Now we fix the elevation angle to be $10°$ and sweep the azimuth angle from $0°$ to $180°$. The total CRB, given by the sum of the CRBs for the elevation and azimuth, is shown in Fig. 6 for both the isotropic and the first directional subarray in Fig. 4(b). The CRB of optimum directional subarrays corresponding to each azimuth angle is also shown for comparison. We can see that reconfiguring optimum directional subarrays adaptively can achieve almost the same estimation performance regardless of the azimuth angle. In other words, the dependence of the estimation variance on arrival angles can be compensated by reconfiguring array geometry, which enables directional subarrays to mimic the angle-independent performance as isotropic subarrays while offering a better estimation performance.

VI. Conclusion

In this letter, we proposed an array thinning strategy for enhancing DOA estimation performance. Problem formulation and solution of antenna selection based on CRB for both isotropic and directional arrays were provided. We presented a Dinklebach-type algorithm and convex relaxation to solve the combinational optimization in polynomial time. For multi-source scenarios, the PSL constrained isotropic array is universal and can be utilised initially for some prior information.

The optimum directional subarray is then reconfigured for each estimated source for performance enhancement.

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