Efficient Two-dimensional Frequency and Damping Estimation by Interpolation on Fourier Coefficients

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***Abstract*—This paper focuses on the efficient estimation of the frequencies and damping factors of a single two-dimensional (2-D) damped complex exponential in additive Gaussian noise. We derive the estimators by extending the FFT-based frequency estimator that relies on interpolation on Fourier coefficients to 2-D damped signals. Performance analysis shows that the algorithm can achieve minimum variances at the fixed point when implemented in an interleaved manner for two iterations. Furthermore, we propose linearised version of the estimators that render them more amenable to real-time DSP implementation. We also demonstrate that the iterative implementation of the algorithm combining both versions is both unbiased and accurate.**

***Index Terms*—Two-dimensional (2-D) parameter estimation, interpolation algorithm, nuclear magnetic resonance (NMR) spectroscopy, zero-padding.**

I. Introduction

TWO-DIMENSIONAL (2-D) parameter estimation is a significant research problem that appears in many engineering applications. In 2-D nuclear magnetic resonance (NMR) spectroscopy, for example, the signal is modeled as a sum of damped complex exponentials in additive noise, [1], [2], and the frequencies, damping factors as well as amplitudes of the signal contain key information on the composition of the chemical sample. An acquired NMR signal usually comprises a large number of time samples, demanding a computationally simple and accurate method to estimate the key parameters. This problem is certainly exacerbated in the 2-D NMR case.

In this paper, we focus on the efficient estimation of the frequencies and damping factors of a single 2-D damped complex exponential in noise. The signal model is given by:

$$x(m,n) = Ae^{(-\eta+j2\pi\mu)m+(-\gamma+j2\pi\nu)n} + w(m,n) \qquad (1)$$

where $m = 0 \ldots M-1$, $n = 0 \ldots N-1$. $A = |A|e^{j\phi}$ represents the complex amplitude of the signal while $\mu, \nu \in [-0.5, 0.5]$ and η, $\gamma > 0$ are respectively the normalized frequencies and damping factors that we need to estimate. The noise terms $w(m,n)$ are assumed to be complex white Gaussian noise with zero mean and variance σ^2. The nominal signal to noise ratio (SNR) is then given by $\rho = |A|^2/\sigma^2$, [2].

Over the past few decades, various high-resolution techniques have been proposed to solve the 2-D frequency/parameter estimation problem. These include approaches such as MEMP [3], 2-D ESPRIT [4] and IMDF [5]. However, these suffer from high computational cost due to their use of the singular value decomposition (SVD) [5]. Recently in [6], So *et al.* presented the PUMA method for single-tone signal estimation and showed that it can achieve the Cramér-Rao lower bound (CRLB). Although this method has a lower computation complexity than high resolution estimators, it still requires the SVD operation, which severely restricts its implementation as the signal size becomes large.

Computationally simple parameter estimators have been developed for the undamped and damped 1-D single tone case, [2], [7], [8]. These operate in the frequency domain and can thus take advantage of the fast Fourier transform (FFT) algorithm to reduce the computational load. In particular, the estimators of [2], [8], achieve a performance that is extremely close to the CRLB. Recent work extended these estimators to the 2-D (undamped) exponentials, [9]. The work presented in this paper builds on [9], by formulating these estimators in the case of a single 2-D damped exponential. Unlike [9], however, the estimators are derived here in the general sense of both an arbitrary interpolation location and an arbitrary amount of zero-padding in each dimension.

The paper is organised as follows. In section II, we present the generalised 2-D parameter estimation algorithm. In Section III, performance analysis is carried out including the analysis of theoretical variances and the linearisation of the estimators. Simulation results are given in Section IV and finally, relevant conclusions is drawn in Section V.

II. Estimation Algorithm

Let $\hat{x}$ be the estimate of a quantity x and consider the case where both dimensions are zero-padded to lengths $K = rM$ and $L = sN$ respectively, where $r, s \geq 1$. The true frequencies of the signal can then be expressed as $\mu = (k_0 + \delta)/K$ and $\nu = (l_0 + \zeta)/L$, where k_0 and l_0 are integer indices and δ and ζ denote the frequency residuals. Ignoring the noise, the coarse estimation stage, with the maximizer applied to the KL-point periodogram, returns the corresponding maximum bin (k_0, l_0), [2]. We then examine the Fourier coefficients at location $(k_0 + p, l_0 + q)$, where $0 \leq p \leq r$ and $0 \leq q \leq s$:

$$\begin{aligned} X_{p,q} &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x(m,n)e^{-j2\pi(m\frac{k_0+p}{K}+n\frac{l_0+q}{L})} \\ &= A\frac{(1-e^{-M\eta+j2\pi\frac{\delta-p}{r}})(1-e^{-N\gamma+j2\pi\frac{\zeta-q}{s}})}{(1-e^{-\eta+j2\pi\frac{\delta-p}{K}})(1-e^{-\gamma+j2\pi\frac{\zeta-q}{L}})}. \end{aligned} \qquad (2)$$

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Denoting $z_h = e^{\eta - j2\pi\frac{\delta}{K}}$, $z_g = e^{\gamma - j2\pi\frac{\zeta}{L}}$ and

$$b_{p,q} = A\frac{(1-e^{-M\eta+j2\pi\frac{\delta-p}{r}})(1-e^{-N\gamma+j2\pi\frac{\zeta-q}{s}})}{1-e^{-\gamma+j2\pi\frac{\zeta-q}{L}}},$$

$$c_{p,q} = A\frac{(1-e^{-M\eta+j2\pi\frac{\delta-p}{r}})(1-e^{-N\gamma+j2\pi\frac{\zeta-q}{s}})}{1-e^{-\eta+j2\pi\frac{\delta-p}{K}}},$$

we have

$$X_{p,q} = \frac{b_{p,q}}{1-z_h^{-1}e^{-j2\pi\frac{p}{K}}} \tag{3}$$

$$= \frac{c_{p,q}}{1-z_g^{-1}e^{-j2\pi\frac{q}{L}}}. \tag{4}$$

Now the coefficient at location (k_0+p-r, l_0+q) is

$$X_{p-r,q} = \frac{b_{p-r,q}}{1-z_h^{-1}e^{-j2\pi\frac{p-r}{K}}}.$$

Using the fact that $b_{p,q} = b_{p-r,q}$, we can recover z_h by:

$$z_h = \frac{e^{-j2\pi\frac{p}{K}}X_{p,q} - e^{-j2\pi\frac{p-r}{K}}X_{p-r,q}}{X_{p,q}-X_{p-r,q}}. \tag{5}$$

Thus the estimators of δ and η are given by:

$$\hat{\delta} = -\frac{K}{2\pi}\mathbf{Im}\{\ln(z_h)\}; \qquad \hat{\eta} = \mathbf{Re}\{\ln(z_h)\}. \tag{6}$$

Similarly, by combining the coefficient

$$X_{p,q-s} = \frac{c_{p,q-s}}{1-z_g^{-1}e^{-j2\pi\frac{q-s}{L}}}$$

with (4), and using the fact $c_{p,q} = c_{p,q-s}$, we can recover z_g by:

$$z_g = \frac{e^{-j2\pi\frac{q}{L}}X_{p,q} - e^{-j2\pi\frac{q-s}{L}}X_{p,q-s}}{X_{p,q}-X_{p,q-s}} \tag{7}$$

and the estimators of ζ and γ become:

$$\hat{\zeta} = -\frac{L}{2\pi}\mathbf{Im}\{\ln(z_g)\}; \qquad \hat{\gamma} = \mathbf{Re}\{\ln(z_g)\}. \tag{8}$$

III. Performance Analysis

In this section, we present the performance analysis of the 2-D estimators proposed above. In the first subsection, we derive the theoretical variances of the estimators, and then find the best interpolation location of each dimension. This is followed by a discussion on the iterative implementation of the algorithm. The linearisation of the estimators is then dealt with in the second subsection.

A. Theoretical Variance Analysis

For brevity, we give the variance derivation of $\hat{\delta}$ and $\hat{\eta}$ as the variances of $\hat{\zeta}$ and $\hat{\gamma}$ can be easily obtained by the appropriate substitutions. In order to simplify the notation, let X_+ and X_- denote $X_{p,q}$ and $X_{p-r,q}$ and also put $\lambda_+ = e^{-j2\pi\frac{p}{K}}$ and $\lambda_- = e^{-j2\pi\frac{p-r}{K}}$. Including the noise terms, (5) becomes:

$$\tilde{z}_h = \frac{\lambda_+(X_++W_+)-\lambda_-(X_-+W_-)}{(X_++W_+)-(X_-+W_-)}$$

$$= \frac{(\lambda_+X_+-\lambda_-X_-)+(\lambda_+W_+-\lambda_-W_-)}{(X_+-X_-)\left(1+\frac{W_+-W_-}{X_+-X_-}\right)}. \tag{9}$$

At high enough SNR, we have that $|W_\pm/X_\pm| \ll 1$, [2]. Therefore, (9) can be written as:

$$\tilde{z}_h \approx \left(z_h + \frac{\lambda_+W_+-\lambda_-W_-}{X_+-X_-}\right)\left(1-\frac{W_+-W_-}{X_+-X_-}\right)$$

$$\approx z_h + \epsilon \tag{10}$$

where ϵ denotes the error:

$$\epsilon = \frac{(\lambda_+-z_h)W_+-(\lambda_--z_h)W_-}{X_+-X_-}.$$

Now we expand $\ln(\tilde{z}_h)$ as a Taylor series about z_h:

$$\ln(\tilde{z}_h) \approx \ln(z_h) + \frac{\epsilon}{z_h} + O\left[\left(\frac{\epsilon}{z_h}\right)^2\right]$$

$$\approx \eta - j\frac{2\pi}{K}\delta + \frac{\epsilon}{z_h}. \tag{11}$$

The approximate expressions of the theoretical variance of $\hat{\mu}$ and $\hat{\eta}$ become

$$\mathrm{var}[\hat{\mu}] \approx \frac{1}{4\pi^2}\mathrm{var}\left[\mathbf{Im}\left\{\frac{\epsilon}{z_h}\right\}\right] \approx \frac{MNQ_1Q_2}{32\pi^2\rho\sin^2\left(\frac{\pi}{M}\right)Q_3}, \tag{12}$$

$$\text{and } \mathrm{var}[\hat{\eta}] \approx \mathrm{var}\left[\mathbf{Re}\left\{\frac{\epsilon}{z_h}\right\}\right] \approx \frac{MNQ_1Q_2}{8\rho\sin^2\left(\frac{\pi}{M}\right)Q_3}, \tag{13}$$

where $\mathrm{var}[\mathbf{Re}\{W_\pm\}] = \mathrm{var}[\mathbf{Im}\{W_\pm\}] = MN\sigma^2/2$ is used here and:

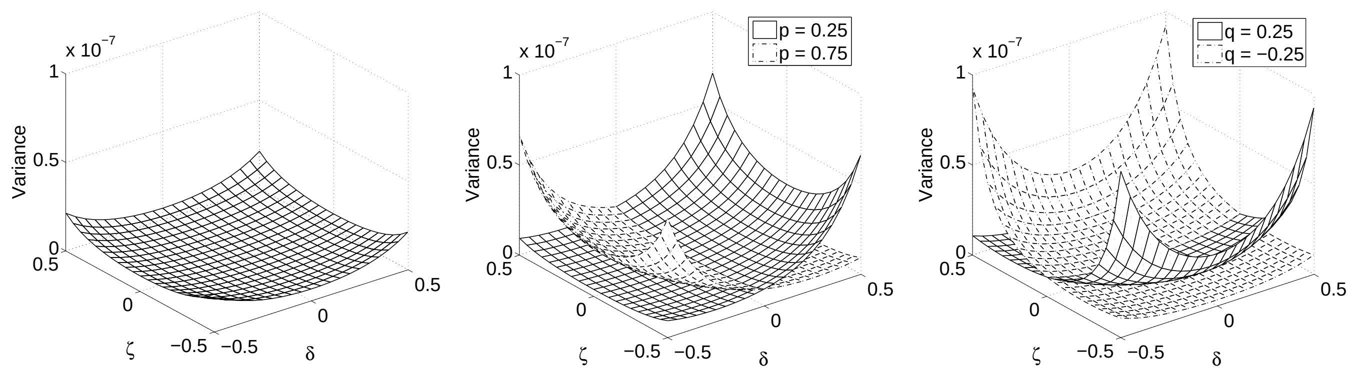
$$Q_1 = \left\{1+e^{-2\eta}-2e^{-\eta}\cos\left[\frac{2\pi}{K}(\delta-p)\right]\right\} \times\left\{1+e^{-2\eta}-2e^{-\eta}\cos\left[\frac{2\pi}{K}(\delta-p+r)\right]\right\} \times\left\{1+e^{-2\gamma}-2e^{-\gamma}\cos\left[\frac{2\pi}{L}(\zeta-q)\right]\right\};$$

$$Q_2 = 2+2e^{-2\eta} -2e^{-\eta}\left\{\cos\left[\frac{2\pi}{K}(\delta-p)\right]+\cos\left[\frac{2\pi}{K}(\delta-p+r)\right]\right\};$$

$$Q_3 = e^{-2\eta}\left\{1+e^{-2M\eta}-2e^{-M\eta}\cos\left[\frac{2\pi}{r}(\delta-p)\right]\right\} \times\left\{1+e^{-2N\gamma}-2e^{-N\gamma}\cos\left[\frac{2\pi}{s}(\zeta-q)\right]\right\}.$$

Applying similar arguments to z_g leads to the theoretical variances of $\hat{\nu}$ and $\hat{\gamma}$. Alternatively, these can also be obtained by appropriate parameter substitutions in Eqs. (12) and (13).

The variance expressions reveal that the values of p and q affect the estimation performance. Therefore, we expect there to be an interpolation location for each dimension that produces the best performance for a zero-padding amounts. To find the best interpolation location of the first dimension, we examine the value of $\mathrm{var}[\hat{\mu}]$ versus δ and ζ under various values of p, q in the unpadded case ($r = s = 1$) when the



(a) Case I (b) Case II (c) Case III

Fig. 1: Theoretical variance of $\hat{\mu}$ under different values of p and q when $r = s = 1$ ($\eta = \gamma = 0.01, M = N = 32, \rho = 10$dB): (a) $p = 0.5, q = 0$; (b) $p = 0.5 \pm 0.25, q = 0$; (c) $p = 0.5, q = \pm 0.25$.

signal size and SNR are fixed. In Fig. 1, we plot the variances obtained by five different sets of p and q with $M = N = 32$, $\eta = \gamma = 0.01$, $\rho = 10$dB. It can be easily observed that setting $p = r/2$ and $q = 0$ centers the variance at $\delta = \zeta = 0$ and gives the smallest maximum values for it. The same conclusion can be reached for z_g and the best interpolation location of $p = 0$ and $q = s/2$ are found for the second dimension. By interpolating on the best location in both dimensions, we can obtain the most robust estimation for all the parameters. From this point on, the estimators are assumed to be working on the best interpolation locations unless otherwise indicated.

The variances of the various quantities are all dependent on the frequency residuals. In order to consistently obtain the lowest estimation variances at the fixed point $\delta = \zeta = 0$, we proceed in line with the methodology proposed in [9] and [2] to consider the alternative and iterative implementation of the proposed estimators. Therefore, the estimation is applied according to the following sequence: $\delta_1 \rightarrow \zeta_1 \rightarrow \delta_2 \rightarrow \zeta_2$ to obtain residual estimates and remove all the previous estimation from the maximum bin before the next estimation. In the second iteration, the damping factors are estimated alongside with the frequency residuals. As a result of the iterative implementation, the estimation variances converge to $\delta = \zeta = 0$ and reach their minimum values consistently.

B. *Linearised Estimators*

The expression of (5) and (7) involve logarithmic and angle operations, which are undesirable in many real-time applications. Therefore, we give alternative estimation expressions that avoid these non-linear functions.

Applying the expansion $e^x = 1 + x + O(x^2)$ to z_h we have

$$z_h = 1 + \eta - j\frac{2\pi}{K}\delta + O\left[\left(\eta - j\frac{2\pi}{K}\delta\right)^2\right]. \quad (14)$$

Thus we can obtain the linearised estimators of δ and η as:

$$\hat{\delta}_L = -\frac{K}{2\pi}\mathbf{Im}\{z_h\}; \qquad \hat{\eta}_L = \mathbf{Re}\{z_h\} - 1. \quad (15)$$

Similarly we can also find the linearised version of (8):

$$\hat{\zeta}_L = -\frac{L}{2\pi}\mathbf{Im}\{z_g\}; \qquad \hat{\gamma}_L = \mathbf{Re}\{z_g\} - 1. \quad (16)$$

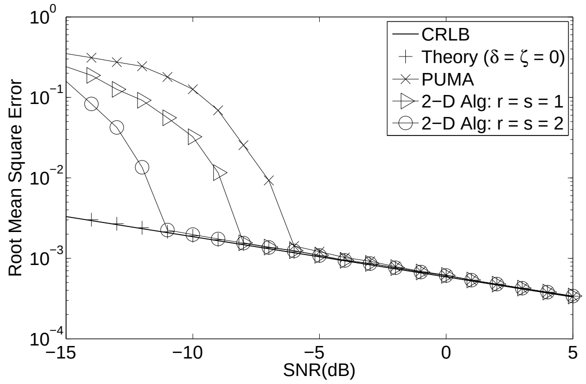
The cost of achieving the computational simplicity of the estimation expressions is that a bias of $\varepsilon \sim O\left[\left(\eta - j\frac{2\pi}{K}\delta\right)^2\right]$ is introduced into the estimators. In order to strike a balance between the computational cost and the estimation performance, we propose the use of the linearised estimators (15) and (16) in the first iteration of the estimation followed by the exact version (5) and (7) in the second iteration. This means that the final estimates are bias free. We now summarize the 2-D parameter estimation algorithm:

1) Zero-pad the signal to $K = rM$, $L = sN$;
2) Calculate $X(k,l) = FFT(x)(x$ being the zero-padded signal) and find the maximum bin $(\hat{k}_0, \hat{l}_0)$ of the periodogram $|X(k,l)|^2$;
3) Use (15) to obtain $\hat{\delta}_1$ and update $(\hat{k}_1, \hat{l}_0) = (\hat{k}_0 + \hat{\delta}_1, \hat{l}_0)$; Then use (16) to obtain $\hat{\zeta}_1$ and set $(\hat{k}_1, \hat{l}_1) = (\hat{k}_1, \hat{l}_0 + \hat{\zeta}_1)$;
4) Use (5) to obtain $\hat{\delta}_2$, $\hat{\eta}$ and update $(\hat{k}_2, \hat{l}_1) = (\hat{k}_1 + \hat{\delta}_2, \hat{l}_1)$; Then use (7) to obtain $\hat{\zeta}_2$, $\hat{\gamma}$ and set $(\hat{k}_2, \hat{l}_2) = (\hat{k}_2, \hat{l}_1 + \hat{\zeta}_2)$;
5) Find $\hat{\mu} = \frac{\hat{k}_2}{K}$ and $\hat{\nu} = \frac{\hat{l}_2}{L}$.

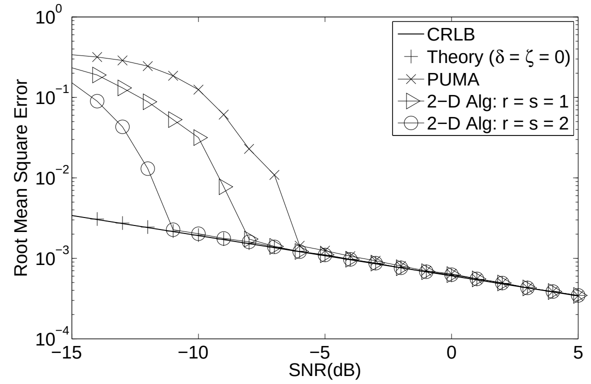
IV. Simulation Results

In this section, we present simulation results in order to verify the performance of the algorithm presented above.

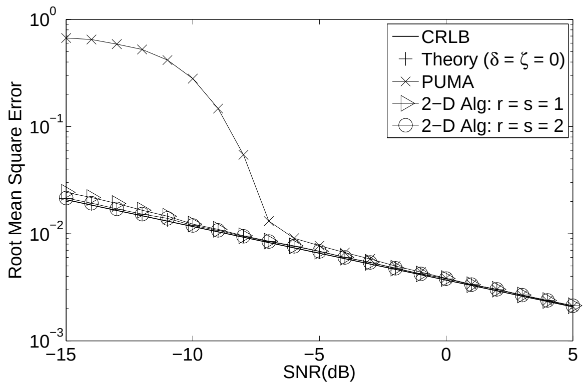
In Fig. 2 we show the root mean square error (RMSE) of both frequency and damping factor estimates versus SNR when $\eta = 0.01$, $\gamma = 0.02$ and $M = N = 32$. The algorithm is simulated in both unpadded case $(r = s = 1)$ and padding both dimensions to twice of their original lengths $(r = s = 2)$. For the sake of comparison, we also show the results estimated by the PUMA algorithm [6], the theoretical variances at the fixed point $\delta = \zeta = 0$ as well as the 2-D CRLB [10]. Frequencies are selected randomly in 5,000 Monte Carlo runs. We can observe that for the unpadded case, the novel algorithm has a breakdown threshold of about -8dB, which performs slightly better than PUMA. This difference is more observable



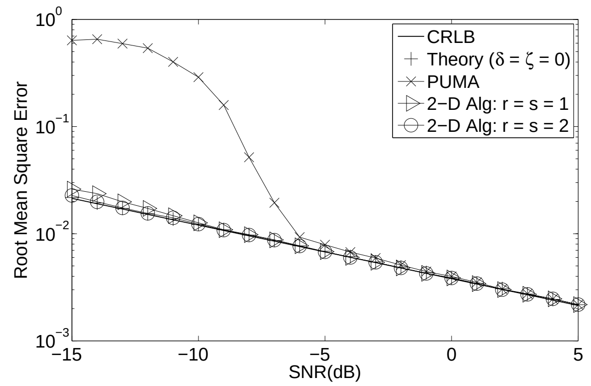
(a) RMSE of $\hat{\mu}$



(b) RMSE of $\hat{\nu}$



(c) RMSE of $\hat{\eta}$



(d) RMSE of $\hat{\gamma}$

Fig. 2: RMSE of different estimators versus SNR when $\eta = 0.01, \gamma = 0.02$ and $M = N = 32$.

in damping factor estimates shown in Fig. 2c and Fig. 2d. The novel estimators show improved robustness after zero-padding having a breakdown threshold of -11dB. This is due to the improved performance of the maximum bin search during the coarse estimation stage. In high SNR, both algorithms share similar performance that is quite close to the CRLB.

V. Conclusion

In this paper, we have proposed a computationally simple algorithm for estimating frequencies and damping factors of a single 2-D damped complex exponential by generalising the interpolation locations of Fourier coefficient as well as the amount of zero-padding. We derived theoretical expressions of the estimation variances and showed that there exists a best interpolation location for each dimension. In order to further reduce the computational burden, we have developed a linearised version of the estimators that avoids the logarithmic and angle operations. By combining the exact version and the linearised version, the iterative implementation of the estimators can converge to the fixed point, where lowest estimation variances can be obtained. Finally, the simulation results are used to verify the theory.

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