Proofs for:

Truncated Hawkes Point Process Modelling: System Theory and System Identification

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A Proof of Result II

We open with a preliminary lemma.

Lemma A1. $\int_0^t p^{(n)}(u)du \le F^n(t)$.

Proof. The proof is by induction. Firstly $\int_0^t p^{(1)}(u)du = \int_0^t p(u)du = F(t)$. Then consider that

$$\begin{aligned}
\int_0^t p^{(n)}(u)du &= \int_0^t \int_0^u p(u-v)p^{(n-1)}(v)dvdu \\
&= \int_0^t p^{(n-1)}(v)[\int_v^t p(u-v)du]dv \\
&= \int_0^t p^{(n-1)}(v)\int_0^{t-v} p(x)dxdv \\
&\le F(t)\int_0^t p^{(n-1)}(v)dv \\
&\le F(t)F^{n-1}(t) = F^n(t) \text{ as required.}
\end{aligned}$$

Proof of Result II. We rewrite the renewal equation and iterate it as follows (dropping the argument for clarity)

$$\begin{aligned}
m &= c + Hp * m \\
&= c + Hp * (c + Hp * m) \\
&= c + cHF(t) + H^2 p^{(2)} * m \\
&= \cdots \\
&= c\sum_0^{n-1} H^r \int_0^t p^{(r)}(u)du + H^n p^{(n)} * m
\end{aligned}$$

We now use this m-identity in two ways. Firstly to show only if and then to show if.

only if. Assume $H \ge 1$.
Noting that $m(t) \ge c > 0$ we have

$$m(t) \ge c\sum_0^n H^r F_r(t) \text{ where } F_r(t) = \int_0^t p^{(r)}(u)du$$

Now we argue by contradiction. Assume $m(t)$ is bounded i.e. $m(t) \le B < \infty$ for all t. Then $B \ge c\sum_0^n H^r F_r(t)$ for all t. It follows that $B \ge c\sum_0^n F_r(t)$ for all t. Now let $t \to \infty$ so that $F_r(t) \to 1$ for each r. Then $B \ge c\sum_0^n 1 \ge cn$. Now let $n \to \infty$ and we deduce $B = \infty$ which is a contradiction as required.

if. Assume $H < 1$. Then using Lemma A1 in the m-identity gives

$$\begin{aligned}
m(t) &\le c\sum_0^{n-1} H^r F^r(t) + H^n(p^{(n)} * m)(t) \\
&= c\frac{1 - H^n F^n(t)}{1 - HF(t)} + H^n(p^{(n)} * m)(t) \\
&\le \frac{c}{1-H} + H^n(p^{(n)} * m)(t)
\end{aligned}$$

Now set $m^*(t) = \sup_{0\le u\le t} m(u)$. Then via Lemma A1

$$\begin{aligned}
(p^{(n)} * m)(t) &= \int_0^t p^{(n)}(t-u)m(u)du \\
&\le m^*(t)\int_0^t p^{(n)}(u)du \le m^*(t)F^n(t)
\end{aligned}$$

Thus

$$\begin{aligned}
m(t) &\le \frac{c}{1-H} + H^n F^n(t)m^*(t) \\
\Rightarrow m^*(t) &\le \frac{c}{1-H} + H^n F^n(t)m^*(t) \\
\Rightarrow m^*(t) &\le \frac{c}{1-H}\frac{1}{1-H^n F^n(t)}
\end{aligned}$$

Now let $n \to \infty$ to get

$$m^*(t) \le \frac{c}{1-H} \Rightarrow m(t) \le \frac{c}{1-H}$$

as required. The proof is complete.

B Calculation of Φ_l

We derive a basic property using integration by parts:

$$\begin{aligned}
\Phi_l(t) &= \int_t^\infty \phi_l(u)du = \int_t^\infty \frac{(u\beta_o)^{l-1}}{(l-1)!}e^{-\beta_o u}\beta_o du \\
&= \int_{t\beta_o}^\infty \frac{v^{l-1}}{(l-1)!}e^{-v}dv \\
&= \frac{v^{l-1}}{(l-1)!}(-e^{-v})|_{t\beta_o}^\infty + \int_{t\beta_o}^\infty \frac{v^{l-2}}{(l-2)!}e^{-v}dv \\
&= \frac{(t\beta_o)^{l-1}}{(l-1)!}e^{-t\beta_o} + \Phi_{l-1}(t) \\
\Rightarrow \quad & \Phi_l(t) = \frac{1}{\beta_o}\phi_l(t) + \Phi_{l-1}(t) \\
&= \Phi_{l-1}(t) - \frac{1}{\beta_o}\dot{\Phi}_l(t)
\end{aligned}$$

Integrating this gives

$$\int_0^T \Phi_l(t)dt = \mathcal{I}_l = \mathcal{I}_{l-1} + \frac{1}{\beta_o}(1 - \Phi_l(T)) \qquad (2.1)$$

since $\Phi_l(0) = 1$. Also, $\mathcal{I}_l(T) < \phi_l(0) = 1$ Note that $\Phi_1(t) = e^{-\beta_o t}$ and so we have

$$\mathcal{I}_1 = \int_0^T e^{-\beta_o t}dt = \frac{1}{\beta_o}(1 - e^{-\beta_o t}) \qquad (2.2)$$

The update for $\Phi_l(T)$ is given by

$$\Phi_l(T) = \Phi_{l-1}(T) + \frac{1}{\beta_o}\phi_l(T) \qquad (2.3)$$

with $\Phi_1(T) = e^{-\beta_o T}$.

C Calculation of B_l for HL

We take $T_o = 0$ so that $x_1(T_o) = x_1(0) = 0$. Denote $N_T = n$ so the final event time is T_n.

We now proceed step by step

C.1 Calculation of B_1

We have

$$\begin{aligned}
x_1(t) &= \Sigma_{j:T_j<t} e^{-\beta_o(t-T_j)}\beta_o \\
&= \sum_{j=1}^n H(t-T_j)e^{-\beta_o(t-T_j)}\beta_o \\
\Rightarrow B_1 &= \int_0^T x_1(t)dt \\
&= \textstyle\sum_1^n \int_0^T H(t-T_j)e^{-\beta_o(t-T_j)}\beta_o \\
&= \textstyle\sum_1^n \int_{T_j}^T e^{-\beta_o(t-T_j)}\beta_o \\
&= \textstyle\sum_1^n \int_0^{T-T_j} e^{-\beta_o v}dv \\
&= \textstyle\sum_1^n [-e^{-\beta_o v}]_0^{T-T_j} \\
&= \textstyle\sum_1^n (1 - e^{-\beta_o(T-T_j)}) \\
&= n - \textstyle\sum_1^n e^{-\beta_o(T-T_j)} = n - \frac{x_1(T)}{\beta_o}
\end{aligned}$$

C.2 Calculation of B_{l+1}

For this case we have

$$\begin{aligned}
x_{l+1}(t) &= \textstyle\sum_1^n H(t-T_j)\phi_l(t-T_j) \\
\Rightarrow B_{l+1} &= \int_0^T x_{l+1}(t)dt \\
&= \textstyle\sum_1^n \int_{T_j}^T \phi_l(t-T_j)dt \\
&= \textstyle\sum_1^n \int_0^{T-T_j} \phi_l(u)du \\
&= \textstyle\sum_1^n (1 - \Phi_l(T-T_j)) \\
&= n - \textstyle\sum_1^n \Phi_l(T-T_j)
\end{aligned}$$

Using the Φ_l update from appendix A we get

$$\begin{aligned}
B_{l+1} &= n - \textstyle\sum_1^n \Phi_{l-1}(T-T_j) - \frac{1}{\beta_o}\sum_1^n \phi_l(T-T_j) \\
&= B_l - \frac{1}{\beta_o}x_{l+1}(T)
\end{aligned}$$

D Calculation of $\frac{d\lambda(u)}{d\beta_o}$

We need to calculate $\frac{d\lambda(u)}{d\tau}$ where $\tau = \begin{bmatrix} \beta_o \\ \theta \end{bmatrix}$, $\theta = \begin{bmatrix} c \\ \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ and

$$\begin{aligned} \lambda(u) &= c + \sum_{l=1}^{p} \alpha_l \hat{x}_l(u) \qquad (4.1) \\ \hat{x}_l(u) &= \hat{\lambda}_e \Phi_l(u) + x_l(u) \end{aligned}$$

So we need to calculate

$$\frac{d\lambda(u)}{d\tau} = [\frac{d\lambda(u)}{d\beta_o}, \frac{d\lambda(u)}{d\theta}] \qquad (4.2)$$

From 4.1, it is clear that

$$\frac{d\lambda(u)}{d\theta} = \begin{bmatrix} 1 \\ \hat{x}_1 \\ \cdots \\ \hat{x}_p \end{bmatrix} \qquad (4.3)$$

$$\frac{d\lambda(u)}{d\beta_o} = \sum_{l=1}^{p} \left(\alpha_l \frac{dx_l(u)}{d\beta_o} + \hat{\lambda}_e \frac{d\Phi_l(u)}{d\beta_o} \right) \qquad (4.4)$$

We have:

$$\begin{aligned} x_l(t) &= \sum_{j:T_j<t} \phi_l(t - T_j) \\ \phi_l(u) &= \frac{e^{-u\beta_o}(\beta_o u)^{j-1}\beta_o}{(j-1)!} \end{aligned}$$

Thus

$$\begin{aligned} \frac{dx_l(t)}{d\beta_o} &= \frac{d}{d\beta_o}\left(\sum_{j:T_j<t} \phi_l(t - T_j) \right) \\ &= \sum_{j:T_j<t} \frac{d}{d\beta_o} \phi_l(t - T_j) \end{aligned}$$

Next

$$\begin{aligned} \frac{d\phi_l(u)}{d\beta_o} &= \frac{d}{d\beta_o}\left(e^{-\beta_o u} \frac{(\beta_o u)^{l-1}\beta_o}{(l-1)!} \right) \\ &= \frac{u^{l-1}}{(l-1)!}\left(e^{-\beta_o u} l \beta_o^{l-1} - u e^{-\beta_o u}\beta_o^l \right) \\ &= \frac{u^{l-1}}{(l-1)!}\left(e^{-\beta_o u} l \beta_o^{l-1} \right) - \frac{u^{l-1}}{(l-1)!} u e^{-\beta_o u}\beta_o^l \\ &= e^{-\beta_o u}\frac{(\beta_o u)^{l-1}}{(l-1)!}\beta_o \left(\frac{l}{\beta_o} \right) - u \frac{u^{l-1}}{(l-1)!} e^{-\beta_o u}\beta_o^l \\ &= \phi_l(u)\frac{l}{\beta_o} - \frac{e^{-\beta_o u}}{(l-1)!} u^{l-1}\beta_o^{l-1}\beta_o u \\ \Rightarrow \frac{d\phi_l(u)}{d\beta_o} &= \phi_l(u)\frac{l}{\beta_o} - \phi_l(u) u \\ &= \frac{l}{\beta_o}\left(\phi_l(u) - \phi_{l+1}(u) \right) \end{aligned}$$

We have then

$$\begin{aligned} &\frac{dx_l(t)}{d\beta_o} \\ &= \sum_{j:T_j<t} \frac{d}{d\beta_o}\phi_l(t - T_j) \\ &= \sum_{j:T_j<t} \frac{l}{\beta_o}\left(\phi_l(t - T_j) - \phi_{l+1}(t - T_j) \right) \\ &= \frac{l}{\beta_o}\left(x_l(t) - x_{l+1}(t) \right) \qquad (4.5) \end{aligned}$$

Continuing we have

$$\frac{d\Phi_l(t)}{d\beta_o} = \int_t^\infty \frac{d\phi_l(v)}{d\beta_o} dv$$

Using 4.5 we have,

$$\begin{aligned} &\frac{d\Phi_l(t)}{d\beta_o} \qquad (4.6) \\ &= \int_t^\infty \left(\phi_l(v)\frac{l}{\beta_o} - \phi_l(v) v \right) dv \\ &= \frac{l}{\beta_o}\int_t^\infty \phi_l(v) dv - \int_t^\infty v\phi_l(v) dv \\ &= \frac{l}{\beta_o}\Phi_l(t) - \left(\frac{t}{\beta_o}\phi_l(t) + \frac{l}{\beta_o}\Phi_l(t) \right) \\ &= -\frac{t}{\beta_o}\phi_l(t) \qquad (4.7) \end{aligned}$$

Placing 4.7, and 4.5, in 4.4, we get finally

$$\begin{aligned} \frac{d\lambda(t)}{d\beta_o} &= -\hat{\lambda}_e \sum_{l=1}^{p} \left(\frac{t}{\beta_o} \right) \alpha_l \phi_l(t) \\ &+ \sum_{l=1}^{p} \alpha_l \frac{l}{\beta_o}\left(x_l(t) - x_{l+1}(t) \right) \end{aligned}$$

E Proof of Convergence in Result III

The convergence follows from lemma E2 below, whose proof requires lemma E1.

Lemma E1. If $0 \leq K(v) \to K$ as $v \to \infty$ then

$$\frac{1}{T}\int_0^T K(v)dv \to K \text{ and } \frac{1}{T}\int_0^T |K(v)-K|dv \to 0$$

Proof. The second result implies the first. Given $\epsilon > 0$ we can find T_ϵ so that for all $v \geq T_\epsilon$ we have $|K(v)-K| < \epsilon$. Then

$$\begin{aligned} d_T &= \frac{1}{T}\int_0^T |K(v)-K|dv \\ &= \frac{1}{T}\int_0^{T_\epsilon} |K(v)-K|dv + \frac{1}{T}\int_{T_\epsilon}^T |K(v)-K|dv \end{aligned}$$

Now 1st term $\to 0$ as $T \to \infty$ and $|2nd - term| \leq \epsilon$. Thus $\limsup d_T \leq \epsilon$. But ϵ is arbitrary and the result follows.

Lemma E2. Suppose $0 \leq K(v) \to K$ as $v \to \infty$ then

$$\frac{1}{T}\int_0^T K(v)m(T-v)dv \to K\lambda_e$$

Proof. We use lemma E1 repeatedly. Since $m(t) \to \lambda_e$ then

$$\frac{K}{T}\int_0^T m(T-v)dv = \frac{K}{T}\int_0^T m(u)du \to K\lambda_e$$

So we need only show

$$d_T = \frac{1}{T}\int_0^T [K(v)-K]m(T-v)dv \to 0$$

Again since $\frac{\lambda_e}{T}\int_0^T (K(v)-K)dv \to 0$ we need only show

$$d_T = \frac{1}{T}\int_0^T (K(v)-K)(m(T-v)-\lambda_e)dv \to 0$$

Result II gives $0 \leq m(u) \leq B < \infty$ for all u. Thus

$$\begin{aligned} |d_T| &\leq \sup_{0 \leq v \leq T} |m(T-v)-\lambda_e| \frac{1}{T}\int_0^T |K(v)-K|dv \\ &\leq (B+\lambda_e)\frac{1}{T}\int_0^T |K(v)-K|dv \to 0 \end{aligned}$$

and the lemma is established.

F Calculation of B_l for BAR

From the BAR subsection we have

$$\delta x_l(t) = N_{t-l\delta+\delta} - N_{t-l\delta} = \int_{t-l\delta}^{t-l\delta+\delta} dN_u$$

By inspection we make the crucial observation that

$$x_l(t) = 0 \text{ for } t \leq l\delta - \delta$$

Thus

$$\begin{aligned} \delta B_l &= \int_0^T \delta x_l(t)dt = \int_{l\delta-\delta}^T \delta x_l(t)dt \\ &= \int_{l\delta-\delta}^T (\int_{t-l\delta}^{t-l\delta+\delta} dN_u)dt \end{aligned}$$

Now change variables to $\tau = t - l\delta + \delta$ to get

$$\begin{aligned} \delta B_l &= \int_0^{T-l\delta+\delta} (\int_{\tau-\delta}^{\tau} dN_u)d\tau \\ &= \int_0^{T-l\delta+\delta} (\int_0^{\tau} dN_u - \int_0^{\tau-\delta} dN_u)d\tau = a_l - b_l \end{aligned}$$

Continuing

$$a_l = \int_0^{T-l\delta+\delta} (\int_0^{\tau} dN_u)d\tau$$

Change integration order to get

$$\begin{aligned} a_l &= \int_0^{T-l\delta+\delta} dN_u(\int_u^{T-l\delta+\delta} d\tau) \\ &= \int_0^{T-l\delta+\delta} dN_u(T-l\delta+\delta-u) \\ &= \Sigma_{T_i<T-l\delta+\delta}(T-l\delta+\delta-T_i) \end{aligned}$$

Next

$$b_l = \int_0^{T-l\delta+\delta} (\int_0^{\tau-\delta} dN_u)d\tau$$

Change the order of integration to get

$$b_l = \int_0^{T-l\delta+\delta} dN_u(\int_{u+\delta}^{T-l\delta+\delta} d\tau)$$

$$
\begin{aligned}
&= \int_0^{T-l\delta+\delta} dN_u(T-l\delta-u) \\
&= \Sigma_{T_i<T-l\delta+\delta}(T-l\delta-T_i) \\
&= \Sigma_{T_i<T-l\delta+\delta}(T-l\delta+\delta-T_i-\delta) = a_l - d_l
\end{aligned}
$$

where

$$d_l = \delta\Sigma_{T_i<T-l\delta+\delta}1 = \delta N_{T-l\delta+\delta}$$

Thus we find

$$\delta B_l = a_l - b_l = a_l - [a_l - d_l] = d_l = \delta N_{T-l\delta+\delta}$$

and the claimed result follows.

G Proof of Result IV

We first calculate $\gamma_+(0)$. From the remark following result IV we get

$$
\begin{aligned}
\gamma_+(0) &= \lambda_e \int_{-\infty}^{\infty} [\bar{g}(j\omega) + \bar{g}(-j\omega) + |\bar{g}(j\omega)|^2] \frac{d\omega}{2\pi} \\
&= 2\lambda_e g(0) + \lambda_e \int_{-\infty}^{\infty} |\bar{g}(j\omega)|^2 \frac{d\omega}{2\pi}
\end{aligned}
$$

To continue we need an expression for $\bar{g}(s)$. From (**??**) we find

$$
\begin{aligned}
\bar{h}(s) &= \textstyle\sum_1^p \alpha_l \int_0^{\infty} e^{-su} \frac{(\beta_o u)^{l-1}}{(l-1)!} e^{-\beta_o u} \beta_o du \\
&= \textstyle\sum_1^p \alpha_l \frac{\beta_o^l}{(\beta_o+s)^l} \int_0^{\infty} \frac{[(\beta_o+s)u]^{l-1}}{(l-1)!} e^{-(\beta_o+s)u} du(\beta_o+s) \\
&= \textstyle\sum_1^p \alpha_l \frac{1}{(\tau_o s+1)^l} \int_0^{\infty} \frac{x^{l-1}}{(l-1)!} e^{-x} dx \\
&= \textstyle\sum_1^p \alpha_l \frac{1}{(\tau_o s+1)^l} = \frac{\alpha(\xi)}{\xi^p}
\end{aligned}
$$

where $\xi = \tau_o s + 1$ and $\alpha(\xi) = \sum_1^p \alpha_l \xi^{p-l}$. It follows that

$$\bar{g}(s) = \frac{\alpha(\xi)/\xi^p}{1-\alpha(\xi)/\xi^p} = \frac{\alpha(\xi)}{\xi^p - \alpha(\xi)}$$

Note that $\bar{g}(s)$ is strictly proper and in view of result IV is stable. Using the Laplace transform initial value theorem we find

$$
\begin{aligned}
g(0) &= \lim_{s\to\infty} s\bar{g}(s) \\
&= \lim_{s\to\infty} s \frac{\sum_1^p \alpha_l (\tau_o s+1)^{p-l}}{(\tau_o s+1)^p - \sum_1^p \alpha_l (\tau_o s+1)^{p-l}} \\
&= \lim_{s\to\infty} s \frac{\sum_1^p \alpha_l/(\tau_o s+1)^l}{1 - \sum_1^p \alpha_l/(\tau_o s+1)^l} = \alpha_1/\tau_o
\end{aligned}
$$

Next

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |\bar{g}(j\omega)|^2 \frac{d\omega}{2\pi} \\
&= \int_{-\infty}^{\infty} \frac{|\alpha(\tau_o j\omega+1)|^2}{|(\tau_o j\omega+1)^p - \alpha(\tau_o j\omega+1)|^2} \frac{d\omega}{2\pi} \\
&= V_g/\tau_o \\
V_g &= \int_{-\infty}^{\infty} \frac{|\alpha(j\omega'+1)|^2}{|(j\omega'+1)^p - \alpha(j\omega'+1)|^2} \frac{d\omega'}{2\pi}
\end{aligned}
$$

Thus we find

$$\gamma_+(0) = \frac{\lambda_e}{\tau_o} \mathcal{J} \text{ where } \mathcal{J} = 2\alpha_1 + V_g$$

where $\mathcal{M}, \mathcal{J}$ do not depend on τ_o but only on memory parameters.

To complete the proof we derive the lower bound.

Using Parseval's theorem, result IV(iv) and the HL-HIR bound we find

$$
\begin{aligned}
V_g = \tau_o \int_0^{\infty} g^2(t) dt &\le \tau_o \frac{B_h H}{(1-H)^2} \\
&= \tau_o \beta_o \frac{H}{(1-H)^2} = \frac{H}{(1-H)^2}
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
\mathcal{J} &\le \alpha_1 + \frac{H}{(1-H)^2} \\
\Rightarrow L &\ge \tau_o \frac{H(2-H)}{(1-H)^2} \frac{1}{[2\alpha_1 + H/(1-H)^2]} \\
&= \tau_o \frac{H(2-H)}{H + 2\alpha_1(1-H)^2}
\end{aligned}
$$