Graphical Basis Partitions

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Abstract

A partition of an integer n is *graphical* if it is the degree sequence of a simple, undirected graph. It is an open question whether the fraction of partitions of n which are graphical approaches 0 as n approaches infinity. A partition π is *basic* if the number of dots in its Ferrers graph is minimum among all partitions with the same rank vector as π.

In this paper, we investigate graphical partitions via basis partitions. We show how to efficiently count and generate graphical basis partitions and how to use them to count graphical partitions. We give empirical evidence which leads us to conjecture that, as n approaches infinity, the fraction of basis partitions of n which are graphical approaches the same limit as the fraction of all partitions of n which are graphical.

1 Introduction

A *partition* of a non-negative integer n is a sequence of positive integers $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$ satisfying $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_s$ and $\pi_1 + \pi_2 + \ldots + \pi_s = n$. Let $P(n)$ be the set of partitions of n, where $P(0)$ contains only the empty partition λ, and let $p(n) = |P(n)|$. The partition $\pi \in P(n)$ is said to be *graphical* if there exists a simple undirected graph with degree sequence π. Since the sum of the degrees of the vertices of a graph equals twice the number of edges, a necessary condition for $\pi \in P(n)$ to be graphical is that n is even. (For convenience, we consider λ to be graphical.) Let $G(n)$ denote the set of graphical partitions of an even integer n, and let $g(n) = |G(n)|$. It is an open question, originally

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posed by H. Wilf, whether $\lim_{n\to\infty} g(n)/p(n) = 0$. The best upper bound known is that $\overline{\lim}_{n\to\infty} g(n)/p(n) \leq 0.25$, due to Rousseau and Ali [11]. Recent methods for efficiently counting and generating $G(n)$ [2, 3] allow $g(n)/p(n)$ to be tabulated, but so far these methods have given no insight into the limiting behavior of the ratio.

Several necessary and sufficient conditions to determine whether an integer sequence $(\pi_1, \ldots, \pi_s)$ is graphical have been proposed in the literature. Seven such criteria have been listed and shown to be equivalent in [13]. Among these, the most well known is the Erdős-Gallai condition stated below:

$$\sum_{i=1}^{k} \pi_i \leq k(k-1) + \sum_{j=k+1}^{n} \min(k, \pi_j), \qquad k = 1, \ldots, s. \tag{1}$$

For a proof, see [8], pp 59-61.

A lesser-known condition is the Nash-Williams condition, which works with the rank vector of the partition. For a partition $\pi = (\pi_1, \ldots, \pi_s)$, the associated *Ferrers diagram* is an array of s rows of dots, where row i has π_i dots and rows are left justified. The *conjugate partition* of π is denoted by $\pi' = (\pi'_1, \ldots, \pi'_t)$, where $t = \pi_1$ and π'_i is the number of dots in the i-th column of the Ferrers diagram of π. The *Durfee square* of π is the largest square sub-array of dots in the Ferrers diagram of π. Let d denote the length of a side of the Durfee square. The *rank vector* of π, defined in [7], is the vector $\mathbf{r} = [r_1, r_2, \ldots, r_d]$ whose entries $r_i = \pi_i - \pi'_i$ are the successive ranks of Atkin [1]. The Nash-Williams condition, necessary and sufficient for π to be graphical, is:

$$\sum_{i=1}^{k} (r_i + 1) \leq 0, \qquad k = 1, \ldots, d. \tag{2}$$

This condition is shown in [11] to be equivalent to the Erdős-Gallai condition.

Since the Nash-Williams condition uses only the rank vector of a partition to determine whether the partition is graphical, it becomes natural to consider families of partitions defined by their rank vectors.

As an example, let $R(n)$ be the set of partitions of n for which all rank vector entries are negative. It was noted by Erdős and Richmond [6] that all partitions in $R(n)$ are graphical and hence $g(n) \geq r(n) = |R(n)|$. They observe that $r(n) = p(n) - p(n-1)$ from a result of Bressoud [5] and that $(p(n) - p(n-1))/p(n) \sim \pi/\sqrt{6n}$, from Roth and Szekeres [10], to conclude that

$$\overline{\lim}_{n\to\infty}\frac{\sqrt{n}\ g(n)}{p(n)} \geq \frac{\pi}{\sqrt{6}}.$$

Although every partition has a unique rank vector, the converse it not true. For example, the partitions $(8,6,4,3,3,3,1)$ and $(5,4,3,3)$ both have rank vector $[1,0,-1]$. Gupta, in [7], gives a one-to-one correspondence between rank vectors and a subclass of partitions which he calls *basis partitions.* Let $\mathbf{r}(\pi)$ be the rank vector of $\pi = (\pi_1,\ldots,\pi_s)$ and let the *weight* of π be $|\pi| = \pi_1 + \ldots + \pi_s$. A partition $\pi \in P(n)$ is *basic* iff

$$|\pi| = \min\{\ |\pi'|\ \mid\ \mathbf{r}(\pi') = \mathbf{r}(\pi)\}.$$

Informally, π is basic if and only if the weight of π is minimum over all partitions with the same rank vector as π. The set $B(n)$ of *basis partitions* is the set of all partitions of n which are basic.

For a partition π with rank vector $\mathbf{r}(\pi) = [r_1,\ldots,r_d]$, define the co-rank vector of π, $\mathbf{c}(\pi) = [c_1,\ldots,c_d]$ by $c_i = -r_i$ for $1 \leq i \leq d$. Then the Nash-Williams condition can be restated as

$$\sum_{i=1}^{k}(c_i - 1) \geq 0, \qquad k = 1,\ldots,d. \tag{3}$$

In Section 2, we survey some results on basis partitions. In Section 3, we develop a recurrence for counting graphical basis partitions and compare the fraction of basis partitions of n which are graphical to the fraction of all partitions of n which are graphical. In Section 4, we present an algorithm to generate graphical basis partitions. The algorithm requires only constant amortized time per partition. In fact, this algorithm, without the test for graphical, is the first constant amortized time algorithm we know of for generating basis partitions. Suggestions for further research follow in Section 5.

2 Results on Basis Partitions

We include here only results on basis partitions which will be required in this paper. For further information on basis partitions, including proofs omitted in this section, see [7, 9].

We focus first on the existence and uniqueness of basis partitions.

Theorem 1 [Gupta] *Among all partitions with the same rank vector* $\mathbf{r} = [r_1, \ldots, r_k]$*, there is just one with minimum weight.*

The following simple test will determine whether a partition is basic.

Lemma 1 *A partition* π *with Durfee square of size* d *is basic if and only if both*

$$\pi_d = d \qquad \text{or} \qquad \pi'_d = d \quad \text{and} \tag{4}$$

$$\text{for } 1 \leq i < d \ : \quad \pi_i = \pi_{i+1} \qquad \text{or} \qquad \pi'_i = \pi'_{i+1}. \tag{5}$$

Gupta [7] notes the following bijection, where $p(n,k)$ denotes the total number of partitions of n into parts of size at most k.

Theorem 2 [Gupta] *Let* $\mathbf{r} = [r_1, \ldots, r_d]$ *and let* π *be the basis partition of* $\mathbf{r}$*. The number of partitions of* n *with rank vector* $\mathbf{r}$ *is* $p(m,d)$ *where* $m = (n - |\pi|)/2$*.*

Let $B(n,d)$ be the set of basic partitions of n which have a rank vector of length d and let $b(n,d) = |B(n,d)|$. A partition can be classified according to the length of its rank vector $\mathbf{r}$ and the weight n_0 of the basis partition associated with $\mathbf{r}$. Combining this with Theorem 2 gives the following.

Corollary 1 *We have*

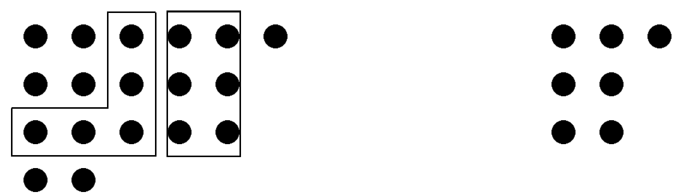
$$p(n) = \sum_{d=0}^{n} \sum_{n_0=0}^{n} b(n_0, d) p((n - n_0)/2,\ d),$$

where $p(n,d) = 0$ *if* n *is not an integer.*

A recurrence for counting $B(n,d)$ is given in [9]:

Theorem 3 *The number* $b(n,d)$ *of basis partitions of* n *with Durfee square of size* d *is: 1, if* $n = d = 0$*; otherwise, 0, if* $n \leq 0$ *or* $d \leq 0$*; and otherwise,*

$$b(n,d) = b(n-d, d) + b(n-2d+1, d-1) + b(n-3d+1, d-1).$$



r = (2,1,2) r = (0,-1)

Figure 1: Deletion of dots in boxes of original partition results in a smaller basis partition

It has been shown in [7] that for a given rank vector, the corresponding basis partition is easy to construct. However, in Section 4, given a rank vector, we will need an explicit formula for the weight of its basis partition in terms of the rank vector elements. We now state and prove such a result:

Theorem 4 *For a rank vector* $\mathbf{r} = [r_1, r_2, \ldots, r_d]$, *the corresponding basis partition* π *has weight* $|\pi| = n(\mathbf{r})$ *where*

$$n(\mathbf{r}) = d^2 + d|r_d| + \sum_{i=1}^{d-1} i|r_i - r_{i+1}|. \tag{6}$$

Proof. Let π be the basis partition corresponding to the given rank vector $\mathbf{r}$. Assume that $r_d \geq 0$ (the other case, $r_d < 0$ can be handled similarly). Then by (4) above, $\pi_d = d + r_d$ and $\pi'_d = d$. Further, by (5) above, either $\pi_{d-1} = d + r_d$ or $\pi'_{d-1} = d$. Now we delete the dots in row d and column d of the Durfee square (a total of $2d - 1$ dots deleted), as also r_d dots from each π_i for $i \leq d$ (another dr_d deleted). In the new Ferrers diagram (see Figure (1)), either row $d-1$ or column $d-1$ has exactly $d-1$ dots. Thus, the new Ferrers diagram satisfies both properties (4) and (5) above with $d-1$ instead of d, and is therefore a basis for the rank vector $\mathbf{r}' = [r_1 - r_d, r_2 - r_d, \ldots, r_{d-1} - r_d]$. This gives us a recursive formula for the number of dots in the Ferrers diagram:

$$n(\mathbf{r}) = 2d - 1 + d|r_d| + n(\mathbf{r}'), \tag{7}$$

which holds similarly in the case that $r_d < 0$. It is easy to verify that the expression in (6) satisfies this recurrence. The recurrence holds for $d \geq 1$. For $d = 0$, $n(\mathbf{r})$ is defined to be zero. $\square$

3 Counting Graphical Basis Partitions

Let $H(n)$ be the set of basis partitions of n which are graphical and let $H(n,d)$ be the set of graphical basis partitions of n with Durfee square of size d. Denote the size of these sets by $h(n)$ and $h(n,d)$, respectively. By the Nash-Williams condition (2), π is a graphical partition of n_0 with rank vector $\mathbf{r}$ if and only if *any* partition of n with rank vector $\mathbf{r}$ is graphical. By Theorem 2, the number of such partitions of n is $p((n-n_0)/2,d)$ where d is the length of $\mathbf{r}$. Thus classifying graphical partitions of n according to the length d of their rank vector and the weight of the corresponding basis partition gives

$$g(n) = \sum_{d=0}^{d=n} g(n,d) = \sum_{d=0}^{n} \sum_{n_0=0}^{n} h(n_0,d)p((n-n_0)/2,\ d).$$

Since $p(n,k)$ is easy to compute, a fast algorithm for computing $h(n,d)$ can be used for efficient computation of $g(n)$.

Let $H(n,d,t,s)$ be the set of basic partitions of n with Durfee square of size d whose co-rank vector $\mathbf{c} = [c_1,\ldots,c_d]$ satisfies:

$$\sum_{i=1}^{k}(c_i - 1 - t) \geq 0, \qquad k = 1,\ldots,d \tag{8}$$

and

$$\sum_{i=1}^{d}(c_i - 1 - t) \geq s. \tag{9}$$

Lemma 2 *For even n, $H(n,d,0,0) = H(n,d)$.*

Proof. When $s = 0$, condition (9) is implied by condition (8) and when $t = 0$, the condition (8) is the Nash-Williams criterion (3) for graphical partitions. □

Let $h(n,d,t,s)$ denote the size of $H(n,d,t,s)$. The recurrence below allows $h(n,d,t,s)$ to be computed within time polynomial in n.

Theorem 5 *For integers n,d,t,s with $s,n,d \geq 0$, $h(n,d,t,s)$ can be defined recursively as follows.*

If $n < d^2$:

$$h(n,d,t,s) = 0.$$

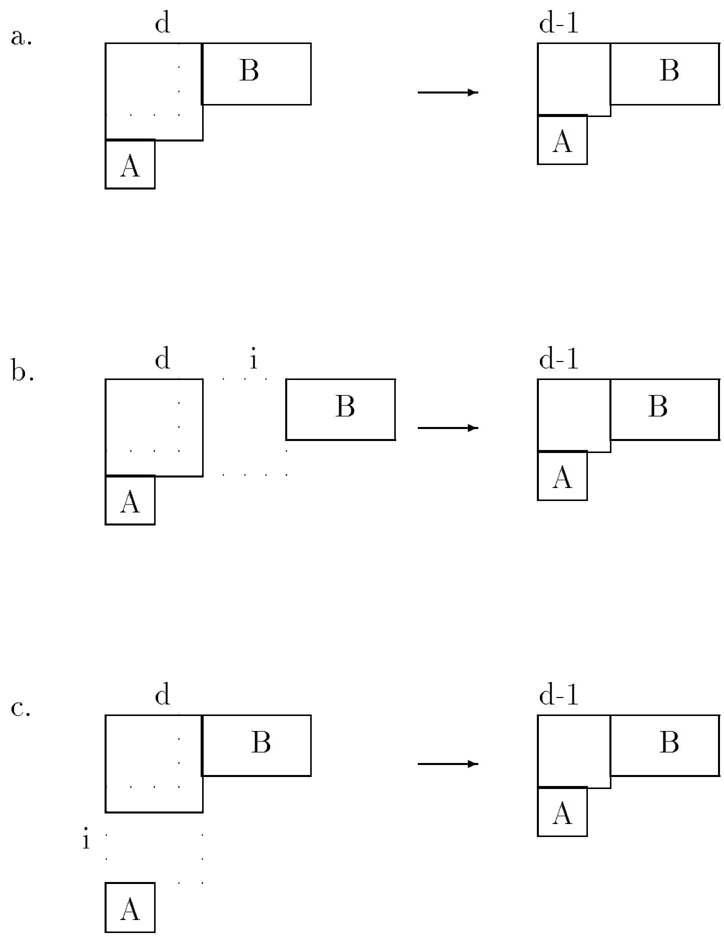


Figure 2: The mapping for cases (a), (b), and (c) in the proof of Theorem 5.

If $n = d^2$*:*

$$h(n,d,t,s) = \begin{cases} 1 & \textit{if } t \le -1 \textit{ and } s \le -d(1+t); \\ 0 & \textit{otherwise.} \end{cases}$$

If $n > d = 1$*:*

$$h(n,d,t,s) = \begin{cases} 2 & \textit{if } n+t+s \le 0; \\ 1 & \textit{if } -n+t+s+2 \le 0; \\ 0 & \textit{otherwise.} \end{cases}$$

If $n > d^2 > 1$*:*

$$h(n,d,t,s) = \sum_{j=-\lfloor (n-d^2)/d \rfloor}^{\lfloor (n-d^2)/d \rfloor} h(n - d(2+|j|)+1,\ d-1,\ t+j,\ \max\{0, s+t+j+1\}).$$

Proof. There are no partitions of n with Durfee square of size d if $n < d^2$. If $n = d^2$, the unique partition π of n with Durfee square of size d has co-rank vector $\mathbf{c} = [0, \ldots, 0]$, so conditions (8) and (9) become

$$0 \ \le \ \sum_{i=1}^{k} (c_i - 1 - t) = -k(1+t), \quad k = 1, \ldots, d$$

and

$$s \ \le \ \sum_{i=1}^{d} (c_i - 1 - t) = -d(1+t).$$

These are satisfied iff $t \le -1$ and $s \le -d(1+t)$.

The only basis partitions of n with Durfee square of size 1 are $\pi = (n)$ and its conjugate $\pi' = (1, 1, \ldots, 1)$. For π, the co-rank vector is $\mathbf{c}(\pi) = [1-n]$ so conditions (8) and (9) become

$$0 \ \le \ c_1 - 1 - t \ = \ 1 - n - 1 - t$$

and

$$s \ \le \ c_1 - 1 - t = \ 1 - n - 1 - t,$$

which are both satisfied iff $n + t + s \le 0$, since $s \ge 0$. For π', the co-rank vector is $\mathbf{c}(\pi') = [n-1]$ so conditions (8) and (9) become $n - 1 - 1 - t \ge 0$ and $n - 1 - 1 - t \ge s$, which are both satisfied iff

$$-n + t + s + 2 \le 0, \tag{10}$$

since $s \geq 0$. Note that if $n+t+s \leq 0$, then since $n \geq 1$,

$$-n+t+s+2 = (n+t+s) - 2(n-1) \leq -2(n-1) \leq 0,$$

so that if $\pi \in H(n,1,s,t)$ then $\pi' \in H(n,1,s,t)$ and therefore $h(n,1,s,t) = 2$ in this case. Otherwise, $H(n,1,s,t)$ contains only π' if (10) holds and is empty if (10) does not hold.

If $n > d^2 > 1$, assume inductively that the theorem holds for any (n', d', t', s') satisfying the hypotheses of the theorem, with $n' < n$.

Let $\pi = (\pi_1, \ldots, \pi_s) \in H(n,d,t,s)$. Since π is basic, by (4) either $\pi_d = d$ or $\pi'_d = d$. So π falls into one of the three mutually exclusive cases (a) $\pi_d = \pi'_d = d$, (b) $\pi_d > d$, or (c) $\pi'_d > d$. Define $\Theta(\pi) = \sigma$, where σ is the partition obtained from the Ferrers diagram of π as follows:

(a) If $\pi_d = \pi'_d = d$: delete row d and column d of π. (See Figure 2(a).)

(b) If $\pi_d > d$: delete row d and columns d through π_d of π. (See Figure 2(b).)

(c) If $\pi'_d > d$: delete rows d through π'_d and column d of π. (See Figure 2(c).)

Note that in all cases, σ has Durfee square of size $d-1$. Further, in case (a), since π is basic, either $\pi_{d-1} = d$ or $\pi'_{d-1} = d$, so that the resulting σ is basic. Similarly, in (b) and (c), either $\pi_{d-1} = \pi_d$ or $\pi'_{d-1} = \pi'_d$, so that $\sigma_{d-1} = d-1$ or $\sigma'_{d-1} = d-1$, so σ will be basic. Let $\mathbf{c} = [c_1, \ldots, c_d]$ be the co-rank vector of π. We show now how to count the partitions π which fall into each of the three cases.

For case (a), $\Theta(\pi) = \sigma \in B(n-2d+1, d-1)$ and $\mathbf{c}(\pi) = [c_1, \ldots, c_{d-1}, 0]$ and $\mathbf{c}(\sigma) = [c_1, \ldots, c_{d-1}]$. Note that $\pi \in H(n,d,t,s)$ if and only if π satisfies conditions (8) and (9) for s and t and this occurs if and only if σ satisfies

$$\sum_{i=1}^{k} (c_i - 1 - t) \geq 0, \qquad k = 1, \ldots, d-1$$

and

$$\sum_{i=1}^{d-1} (c_i - 1 - t) \geq s - c_d + 1 + t \geq s + 1 + t,$$

that is, if and only if $\sigma \in H(n-2d+1, d-1, t, \max\{0, s+t+1\})$.

For case (b), let $j = \pi_d - d$. Then $\Theta(\pi) = \sigma \in B(n - d(2+j) + 1, d-1)$ and $\mathbf{c}(\pi) = [c_1, c_2, \ldots, c_{d-1}, -j]$ and $\mathbf{c}(\sigma) = [c_1 + j, c_2 + j, \ldots, c_{d-1} + j]$. As in case (a), it can be checked that $\pi \in H(n,d,t,s)$ if and only if $\sigma \in B(n - d(2+j) + 1, d-1, t+j, \max\{0, s+1+t+j)\}$.

For case (c), let $j = \pi'_d - d$. Then $\Theta(\pi) = \sigma \in B(n - d(2+j) + 1, d-1)$ and $\mathbf{c}(\pi) = [c_1, c_2, \ldots, c_{d-1}, j]$ and $\mathbf{c}(\sigma) = [c_1 - j, c_2 - j, \ldots, c_{d-1} - j]$. As in case (a), it can be checked that $\pi \in H(n,d,t,s)$ if and only if $\sigma \in B(n - d(2+j) + 1, d-1, t-j, \max\{0, s+1+t-j\})$.

Combining the contributions from cases (a), (b), and (c), respectively, we get

$$\begin{aligned} h(n,d,t,s) &= h(n-2d+1, d-1, t, \max\{0, s+t+1\}) && (11) \\ &+ \sum_{j\geq 1} h(n - d(2+j) + 1, d-1, t+j, \max\{0, s+1+t+j\}) && (12) \\ &+ \sum_{j\geq 1} h(n - d(2+j) + 1, d-1, t-j, \max\{0, s+1+t-j\}). && (13) \end{aligned}$$

To get an upper bound on j, note that $n \geq d^2 + dj$, so that $j \leq \lfloor (n-d^2)/d \rfloor$. Then the right-hand-side terms, (11), (12), and (13) can be combined as

$$h(n,d,t,s) = \sum_{j=-\lfloor (n-d^2)/d \rfloor}^{j=\lfloor (n-d^2)/d \rfloor} h(n - d(2+|j|) + 1, d-1, t+j, \max\{0, s+1+t+j\}).$$

□

With the recurrence of Theorem 5, the number of graphical basis partitions of n,

$$h(n) = \sum_{d=0}^{\lfloor \sqrt{n} \rfloor} h(n,d) = \sum_{d=0}^{\lfloor \sqrt{n} \rfloor} h(n,d,0,0),$$

can be computed using a 4-dimensional table for the values $h(n,d,t,s)$. Each entry can be computed in $O(n)$ time, thereby giving an algorithm which is polynomial in n, even though $h(n)$ appears to grow exponentially. Although the amount of storage can be reduced to $\Theta(n^3)$ by keeping entries only for the two most recent values of d, the storage became prohibitive for us at just over $n = 200$. The values obtained by implementing the recursion of Theorem 5, for even $n \leq 200$, are shown in Tables 1 and 3 at the end of this paper, along with the ratios $h(n)/b(n)$, showing the fraction of basis partitions of n which are graphical. The surprising observation is that this ratio *appears to be approaching the same limit* as $g(n)/p(n)$, the fraction of all partitions which are graphical. These values are given for comparison in Tables 2 and 4. Both ratios appear to be non-increasing for n sufficiently large and we conjecture that the limits exist and:

$$\lim_{n\to\infty} \frac{h(n)}{b(n)} = \lim_{n\to\infty} \frac{g(n)}{p(n)}.$$

4 Generating Basis and Graphical Basis Partitions

In this section we give an efficient algorithm to generate $H(n)$ and prove that the algorithm works in constant amortized time per graphical basis partition. We first show how to efficiently generate basis partitions of n, $B(n)$, and then modify the algorithm to generate $H(n)$.

Since there is a one-to-one correspondence between basis partitions of n and rank vectors whose corresponding basis partition has weight n, we can represent a basis partition by its rank vector. We found it more natural to work with the co-rank vector, which is the negative of the rank vector. It follows then from Theorem 4 that for a co-rank vector $\mathbf{c} = [c_1, c_2, \ldots, c_d]$, the corresponding basis partition π has weight $|\pi| = n(\mathbf{c})$ where

$$n(\mathbf{c}) = d^2 + \sum_{i=1}^{d-1} i|c_i - c_{i+1}| + d|c_d| \tag{14}$$

Since $\mathbf{c}(\pi) = \mathbf{r}(\pi')$, we get the following from (14).

Corollary 2 *For a co-rank vector* $\mathbf{c} = [c_1, c_2, \ldots, c_d]$,

$$\mathbf{n}([c_1, c_2, \ldots, c_d]) - \mathbf{n}([c_2, \ldots, c_d]) = 2d - 1 + p,$$

where

$$p = |c_1 - c_2| + |c_2 - c_3| + \cdots + |c_{d-1} - c_d| + |c_d| \tag{15}$$

The algorithm of this section generates graphical co-rank vectors for a given n, that is, co-rank vectors whose associated basis partitions have weight n and are graphical. It works by successively prepending entries to partially constructed co-rank vectors. If $\Phi = (\phi_1, \phi_2, \ldots, \phi_{d-1})$ is a co-rank vector, then the weight of the basis corresponding to Φ is given by (14). For any integer x, let $\Phi^x = (x + \phi_1) \cdot \Phi$ be the co-rank vector obtained by prepending $\phi_1 + x$ to Φ. By (15), the difference in weights, $n(\Phi^x) - n(\Phi)$ is given by $2d - 1 + |x| + p$, where $p = |\phi_1 - \phi_2| + |\phi_2 - \phi_3| + \ldots + |\phi_{d-2} - \phi_{d-1}| + |\phi_{d-1}|$. Let $B(\Phi, n)$ be the set of co-rank vectors (of any possible length) with weight $n(\Phi) + n$, having suffix Φ.

In the Ferrers diagram, we refer to the L-shaped figure formed by row 1 and column 1 as the *outermost right angle*. Then the intuition for construction of $B(\Phi, n)$ is as follows. The

Ferrers diagram corresponding to the partially constructed co-rank vector Φ has $n(\Phi)$ dots and n more dots are to be added as successive outermost right angles. Then $B(\Phi, 0) = \{\Phi\}$ and for $n > 0$, $B(\Phi, n)$ can be written as the disjoint union,

$$\begin{aligned} B(\Phi, n) &= \bigcup_{|x| \leq n-2d+1-p} B(\Phi^x, n-2d+1-|x|-p) & (16) \\ &= \bigcup_{|x| = n-2d+1-p} B(\Phi^x, 0) \quad \bigcup \\ & \quad \bigcup_{|x| < n-2d+1-p} B(\Phi^x, n-2d+1-|x|-p). & (17) \end{aligned}$$

In each step, the scheme increases the size of the co-rank vector by one, i.e. adds a new outermost right angle to the Ferrers diagram, consuming some or all of the n available dots. The first term considers the case when all the dots are consumed and the second term takes care of the case when Φ^x has to be further augmented to consume the remaining dots. Note that each set in the first term is non-empty if the left hand side is non-empty.

From Corollary 2 it follows that prepending $x + \phi_1$ to the co-rank vector consumes $2d-1+|x|+p$ dots. Adding a new outermost right angle would require at least $2d+1+|x|+p$ remaining dots. Therefore each set in the second term on the right side of (17) will be non-empty if and only if

$$\begin{aligned} n-2d+1-|x|-p &\geq 2d+1+|x|+p \\ \iff \qquad |x| &\leq n/2-2d-p. \end{aligned}$$

Equation (17) can be rewritten as

$$\begin{aligned} B(\Phi, n) = & \bigcup_{|x| = n-2d+1-p} B((x+\phi_1)\cdot\Phi, 0) \quad \bigcup \\ & \bigcup_{|x| \leq n/2-2d-p} B((x+\phi_1)\cdot\Phi, n-2d+1-|x|-p). \qquad (18) \end{aligned}$$

In (18), each set on the right-hand side is non-empty, and therefore in the recursion tree based on (18), there is a one-to-one correspondence between leaves and basis partitions in $B(\Phi, n)$. To generate graphical basis partitions, the tree needs to be pruned.

For a vector $\mathbf{v} = [v_1, v_2, \ldots, v_k]$, define *residual*$(\mathbf{v}) = \sum_{i=1}^{k} (v_i - 1)$, and *need*$(\mathbf{v})$ as the minimum non-negative integer ℓ satisfying,

$$\ell + \sum_{i=1}^{j} (v_i - 1) \geq 0, \qquad j = 1, 2, \ldots, k.$$

Note that by the Nash-Williams condition, (3), $\mathbf{v}$ is a graphical co-rank vector if and only if $need(\mathbf{v}) = 0$.

Denote $need(\Phi)$ by s, and define $H(\Phi, n, s)$ to be the set of co-rank vectors in $B(\Phi, n)$ which are graphical. Note that

$$need((x + \phi_1) \cdot \Phi) = \max\{0, s + 1 - (\phi_1 + x)\},$$

so that we can directly adapt recurrence (18):

$$\begin{aligned} H(\Phi, n, s) \quad = \quad & \bigcup_{|x| = n-2d+1-p} H((x + \phi_1) \cdot \Phi, 0, \max\{0, s + 1 - (\phi_1 + x)\}) \quad \bigcup \\ & \bigcup_{|x| \leq n/2-2d-p} H((x + \phi_1) \cdot \Phi, n - 2d + 1 - |x| - p, \max\{0, s + 1 - (\phi_1 + x)\}) \end{aligned} \tag{19}$$

for which the base case is given by,

$$H(\Phi, 0, s) \quad = \quad \begin{cases} \{\Phi\} & \text{if } s = 0 \\ \{\} & \text{otherwise} \end{cases} \tag{20}$$

Clearly, $H(n) = H(\lambda, n, 0)$. Thus, the recursion of (19) can be used to generate $H(n)$. In the remainder of this section, we show how to implement the recursion efficiently, so that the total time spent is $O(|H(n)|) = O(h(n))$, disregarding the output.

For the vector $\Phi = (\phi_1, \phi_2, \ldots, \phi_{d-1})$, define $p = |\phi_1 - \phi_2| + |\phi_2 - \phi_3| + \ldots + |\phi_{d-2} - \phi_{d-1}| + |\phi_{d-1}|$. Then the following hold.

Lemma 3 *For all $\Psi \cdot \Phi \in B(\Phi, n)$,* residual$(\Psi)$ *is maximized when $\Psi = (\phi_1 + x)$ where $x = n - 2d + 1 - p$.*

Proof. By Corollary 2, the outermost right angle in the Ferrers diagram corresponding to Φ has $2(d-1) - 1 + p$ dots. To add a new outermost right angle, at least $2d - 1 + p$ dots are needed. The residual will be maximized if after adding these dots, the remaining $x = n - 2d + 1 - p$ dots are added to the first column, in which case $\Psi = (\phi_1 + x)$. □

Lemma 4 *$H(\Phi, n, s)$ is non-empty if and only if $\phi_1 + x - 1 \geq s$, where $x = n - 2d + 1 - p$.*

Proof. For all $\Psi \cdot \Phi \in B(\Phi, n)$, if the maximum *residual*(Ψ) (obtained by setting $x = n - 2d + 1 - p$ (from Lemma 3)) is less than $s = $ *need*(Φ), then $H(\Phi, n, s)$ is empty. If however $\phi_1 + x - 1 \geq s$, then $\Psi \cdot \Phi \in H(\Phi, n, s)$ where $\Psi = (\phi_1 + x)$ and hence $H(\Phi, n, s)$ is non-empty. □

Lemma 5 *On the right side of equation (19), if $H((x+\phi_1)\cdot\Phi, n-2d+1-|x|-p, \max(0, s+1-(\phi_1+x)))$ is empty for $x = m$ then it is empty for all $x < m$.*

Proof. Let $s' = \max(0, s+1-(\phi_1+x))$ and $n'_x = n-2d+1-|x|-p$. If $s+1-(\phi_1+m) < 0$ then $\phi_1 + m > 1$ (since s is non-negative). Therefore prepending another element to the co-rank vector with as large a value as possible (by choosing the appropriate value of x as indicated in Lemma 4) will yield a graphical vector, contradicting our assumption that $H(\Phi, n, s)$ is empty. Hence $s' = s + 1 - (\phi_1 + m)$.

If $H((m + \phi_1) \cdot \Phi, n'_m, s')$ is empty, then by 4, $(\phi_1 + m) + (n'_m - 2(d+1) + 1 - (p + |m|)) - 1 < s'$. Substituting values of s' and n'_m gives $2(m - |m|) + \rho < s$, where ρ is independent of m. Since, $x - |x| \leq m - |m|$ for $x < m$, $2(x - |x|) + \rho < s$ for $x < m$, which implies that $(\phi_1 + x) + (n'_x - 2(d+1) + 1 - (p + |x|)) - 1 < s'$. Therefore by Lemma 4, $H((x + \phi_1) \cdot \Phi, n - 2d + 1 - |x| - p, s')$ is empty for $x < m$. □

Based on the above results we present an algorithm in Figure 3 which, as we will prove, generates all elements of $H(n)$ in total time $O(h(n))$. In each step of the recursion, the algorithm prepends an element, $x + \phi_1$ to the partially constructed co-rank vector Φ. It considers the values of x in descending order, in the range as indicated in recursion (19). If a particular value of $x = m$ generates an empty set, subsequent values of x need not be checked (by Lemma 5).

Claim 1 *In the recursion tree of the above algorithm,*

- *the total number of leaves is at most twice the number of graphical basis partitions of n, and*
- *every node with one child has a sibling.*

1. int generate(Φ, n, d, p, s)
   /* Generates rank vectors of $n(\Phi) + n$ with suffix $\Phi = (\phi_1, \phi_2, \ldots, \phi_{d-1})$ and
   $p = |\phi_1 - \phi_2| + |\phi_2 - \phi_3| + \ldots + |\phi_{d-2} - \phi_{d-1}| + |\phi_{d-1}|$ */
2. begin
3.   if (n=0) then output(Φ)
4.   if $((\phi_1 + n - 2d + 1 - p) - 1 < s)$ return 0;
5.   tmp1 $= n - 2d + 1 - p$; $x =$ tmp1
6.   proceed = generate($(\phi_1 + x) \cdot \Phi, 0, d + 1, p + |x|, \max(0, s + 1 - (\phi_1 + x))$)
7.   if (proceed == 0) return 0;
8.   tmp2 = min($n/2 - 2d - p$,tmp1-1)
9.   for (x =tmp2; $x \geq$ -tmp2; $x - -$)
10.   proceed = generate($(\phi_1 + x) \cdot \Phi, n - 2d + 1 - |x| - p, d + 1, p + |x|,$
                    $\max(0, s + 1 - (\phi_1 + x))$)
11.   if (proceed==0) return 1;
12.   x = -tmp1
13.   if $((\phi_1 + x - 1 \geq s)$ and $(x \neq 0)$
14.   proceed = generate($(\phi_1 + x) \cdot \Phi, 0, d + 1, p + |x|, \max(0, s + 1 - (\phi_1 + x))$)
15.   return 1
16. end

Figure 3: Algorithm for generating $H(n)$

Proof. If a node is not a leaf, its leftmost child generates a valid object (follows from Lemma 4 and Step 4 of Figure 3). Further, at most one of its children can be a leaf which fails to generate a valid object (follows from Lemma 5 and the use of variable "proceed" in the algorithm). Thus for every "failure" leaf there is a corresponding "good" leaf. Therefore the number of leaves $\leq 2 \times$ number of objects.

For the second claim, note that a node u, if it is the only child of its parent v, has to be the child corresponding to the call in step 6. Since u was called with $n = 0$, it cannot have any children. Therefore any node u without a sibling cannot have a child, and so every node with a child has a sibling. $\square$

From the above claim, we conclude that the number of nodes is bounded by a constant times the number of leaves which is $O(h(n))$. Moreover, each node involves a constant amount of work. Therefore we conclude that the algorithm, if we exclude time to output the results, averages constant time per item generated.

5 Directions for Further Research

The open question remains as to whether $g(n)/p(n)$ approaches 0, as well as our new conjecture that the fraction of basis partitions which are graphical approaches the same limit as the fraction of all partitions which are graphical. Perhaps generating functions can be found for these quantities which would give insight into their asymptotic behaviour, although they do not seem to be easily derivable from the recurrences here in Theorem 5 or in [2].

In order to be able to collect data for larger values of n, we need faster ways to count (ordinary or basic) graphical partitions, for example, by asymptotically reducing the storage requirements.

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n	$h(n)$	$b(n)$	$h(n)/b(n)$
2	1	2	0.500000
4	1	3	0.333333
6	3	6	0.500000
8	4	10	0.400000
10	6	16	0.375000
12	11	26	0.423077
14	16	40	0.400000
16	23	61	0.377049
18	36	90	0.400000
20	52	130	0.400000
22	71	186	0.381720
24	103	264	0.390152
26	141	370	0.381081
28	197	512	0.384766
30	272	702	0.387464
32	366	952	0.384454
34	482	1282	0.375975
36	657	1715	0.383090
38	863	2278	0.378841
40	1140	3008	0.378989
42	1489	3948	0.377153
44	1951	5150	0.378835
46	2511	6684	0.375673
48	3241	8632	0.375463
50	4155	11094	0.374527
52	5317	14198	0.374489
54	6782	18096	0.374779
56	8574	22972	0.373237
58	10786	29054	0.371240
60	13645	36616	0.372651
62	17111	45984	0.372108
64	21313	57561	0.370268
66	26631	71828	0.370761
68	33020	89358	0.369525
70	41005	110850	0.369914
72	50640	137134	0.369274
74	62373	169196	0.368643
76	76510	208226	0.367437
78	94089	255632	0.368064
80	114991	313082	0.367287
82	140376	382568	0.366931
84	170970	466442	0.366541
86	207837	567482	0.366244
88	251552	688982	0.365107
90	305342	834822	0.365757
92	368474	1009562	0.364984
94	444360	1218584	0.364653
96	534692	1468202	0.364181
98	642593	1765812	0.363908
100	770278	2120101	0.363321

Table 1: Fraction of basis partitions of n which are graphical ($2 \leq n \leq 100$.)

n	$g(n)$	$p(n)$	$g(n)/p(n)$
2	1	2	0.500000
4	2	5	0.400000
6	5	11	0.454545
8	9	22	0.409091
10	17	42	0.404762
12	31	77	0.402597
14	54	135	0.400000
16	90	231	0.389610
18	151	385	0.392208
20	244	627	0.389155
22	387	1002	0.386228
24	607	1575	0.385397
26	933	2436	0.383005
28	1420	3718	0.381926
30	2136	5604	0.381156
32	3173	8349	0.380046
34	4657	12310	0.378310
36	6799	17977	0.378205
38	9803	26015	0.376821
40	14048	37338	0.376239
42	19956	53174	0.375296
44	28179	75175	0.374845
46	39467	105558	0.373889
48	54996	147273	0.373429
50	76104	204226	0.372646
52	104802	281589	0.372181
54	143481	386155	0.371563
56	195485	526823	0.371064
58	264941	715220	0.370433
60	357635	966467	0.370044
62	480408	1300156	0.369500
64	642723	1741630	0.369035
66	856398	2323520	0.368578
68	1136715	3087735	0.368139
70	1503172	4087968	0.367706
72	1980785	5392783	0.367303
74	2601057	7089500	0.366889
76	3404301	9289091	0.366484
78	4441779	12132164	0.366116
80	5777292	15796476	0.365733
82	7492373	20506255	0.365370
84	9688780	26543660	0.365013
86	12494653	34262962	0.364669
88	16069159	44108109	0.364313
90	20614755	56634173	0.363999
92	26377657	72533807	0.363660
94	33671320	92669720	0.363348
96	42878858	118114304	0.363028
98	54481054	150198136	0.362728
100	69065657	190569292	0.362418

Table 2: Fraction of all partitions of n which are graphical ($2 \leq n \leq 100$.)

n	$h(n)$	$b(n)$	$h(n)/b(n)$
102	923765	2541220	0.363512
104	1103815	3041024	0.362975
106	1317309	3633378	0.362558
108	1570056	4334430	0.362229
110	1868705	5162980	0.361943
112	2220359	6140928	0.361567
114	2636855	7293708	0.361525
116	3123822	8650838	0.361101
118	3695909	10246586	0.360697
120	4370543	12120636	0.360587
122	5157648	14318904	0.360199
124	6078890	16894530	0.359814
126	7162034	19908882	0.359741
128	8423730	23432770	0.359485
130	9892375	27547902	0.359097
132	11611088	32348388	0.358939
134	13604431	37942542	0.358553
136	15930339	44455002	0.358348
138	18632427	52028968	0.358116
140	21773062	60828854	0.357940
142	25403221	71043360	0.357573
144	29631976	82888787	0.357491
146	34503277	96612898	0.357129
148	40148343	112499372	0.356876
150	46677794	130872654	0.356666
152	54226543	152103550	0.356511
154	62912051	176615666	0.356209
156	72959764	204892444	0.356088
158	84487617	237485240	0.355759
160	97792760	275022576	0.355581
162	113085815	318220286	0.355370
164	130664848	367893132	0.355171
166	150806514	424968022	0.354866
168	174031171	490498578	0.354805
170	200555761	565681736	0.354538
172	230974979	651876550	0.354323
174	265814962	750625012	0.354125
176	305698161	863675644	0.353950
178	351221683	993010158	0.353694
180	403374047	1140873028	0.353566
182	462791268	1309804878	0.353328
184	530686352	1502680138	0.353160
186	608114168	1722748828	0.352991
188	696298258	1973683620	0.352791
190	796684626	2259632792	0.352573
192	911155119	2585278998	0.352440
194	1041143947	2955905348	0.352225
196	1188972606	3377469559	0.352031
198	1357105119	3856686288	0.351884
200	1547954890	4401119512	0.351718

Table 3: Fraction of basis partitions of n which are graphical ($102 \le n \le 200$.)

n	$g(n)$	$p(n)$	$g(n)/p(n)$
102	87370195	241265379	0.362133
104	110287904	304801365	0.361835
106	138937246	384276336	0.361556
108	174675809	483502844	0.361272
110	219186741	607163746	0.361001
112	274512656	761002156	0.360725
114	343181668	952050665	0.360466
116	428244215	1188908248	0.360200
118	533464959	1482074143	0.359945
120	663394137	1844349560	0.359690
122	823598382	2291320912	0.359443
124	1020807584	2841940500	0.359194
126	1263243192	3519222692	0.358955
128	1560795436	4351078600	0.358715
130	1925513465	5371315400	0.358481
132	2371901882	6620830889	0.358248
134	2917523822	8149040695	0.358021
136	3583515700	10015581680	0.357794
138	4395408234	12292341831	0.357573
140	5383833857	15065878135	0.357353
142	6585699894	18440293320	0.357136
144	8045274746	22540654445	0.356923
146	9815656018	27517052599	0.356711
148	11960467332	33549419497	0.356503
150	14555902348	40853235313	0.356297
152	17692990183	49686288421	0.356094
154	21480510518	60356673280	0.355893
156	26048320019	73232243759	0.355695
158	31551087790	88751778802	0.355498
160	38173235010	107438159466	0.355304
162	46134037871	129913904637	0.355112
164	55694314567	156919475295	0.354923
166	67163674478	189334822579	0.354735
168	80909973315	228204732751	0.354550
170	97368672089	274768617130	0.354366
172	117056456152	330495499613	0.354185
174	140584220188	397125074750	0.354005
176	168675124141	476715857290	0.353827
178	202182888436	571701605655	0.353651
180	242116891036	684957390936	0.353477
182	289666252014	819876908323	0.353305
184	346234896845	980462880430	0.353134
186	413474657328	1171432692373	0.352965
188	493331835384	1398341745571	0.352700
190	588093594457	1667727404093	0.352632
192	700451190712	1987276856363	0.352468
194	833561537987	2366022741845	0.352305
196	991134281267	2814570987591	0.352144
198	1177516049387	3345365983698	0.351984
200	1397805210533	3972999029388	0.351826

Table 4: Fraction of all partitions of n which are graphical ($102 \le n \le 200$.)