

Proofs for:

## Truncated Hawkes Point Process System Identification

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## Modelling: System Theory and

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### A Proof of Result II

We open with a preliminary lemma.

**Lemma A1.**  $\int_0^t p^{(n)}(u)du \leq F^n(t)$ .

*Proof.* The proof is by induction. Firstly  $\int_0^t p^{(1)}(u)du = \int_0^t p(u)du = F(t)$ . Then consider that

$$\begin{aligned} \int_0^t p^{(n)}(u)du &= \int_0^t \int_0^u p(u-v)p^{(n-1)}(v)dvdu \\ &= \int_0^t p^{(n-1)}(v) \left[ \int_v^t p(u-v)du \right] dv \\ &= \int_0^t p^{(n-1)}(v) \int_0^{t-v} p(x)dx dv \\ &\leq F(t) \int_0^t p^{(n-1)}(v)dv \\ &\leq F(t)F^{n-1}(t) = F^n(t) \text{ as required.} \end{aligned}$$

*Proof of Result II.* We rewrite the renewal equation and iterate it as follows (dropping the argument for clarity)

$$\begin{aligned} m &= c + Hp * m \\ &= c + Hp * (c + Hp * m) \\ &= c + cHF(t) + H^2p^{(2)} * m \\ &= \dots \\ &= c \sum_0^{n-1} H^r \int_0^t p^{(r)}(u)du + H^n p^{(n)} * m \end{aligned}$$

We now use this m-identity in two ways. Firstly to show only if and then to show if.

only if. Assume  $H \geq 1$ .

Noting that  $m(t) \geq c > 0$  we have

$$m(t) \geq c \sum_0^n H^r F_r(t) \text{ where } F_r(t) = \int_0^t p^{(r)}(u)du$$

Now we argue by contradiction. Assume  $m(t)$  is bounded i.e.  $m(t) \leq B < \infty$  for all  $t$ . Then  $B \geq c \sum_0^n H^r F_r(t)$  for all  $t$ . It follows that  $B \geq c \sum_0^n F_r(t)$  for all  $t$ . Now let  $t \rightarrow \infty$  so that  $F_r(t) \rightarrow 1$  for each  $r$ . Then  $B \geq c \sum_0^n 1 \geq cn$ . Now let  $n \rightarrow \infty$  and we deduce  $B = \infty$  which is a contradiction as required.

if. Assume  $H < 1$ . Then using Lemma A1 in the m-identity gives

$$\begin{aligned} m(t) &\leq c \sum_0^{n-1} H^r F^r(t) + H^n (p^{(n)} * m)(t) \\ &= c \frac{1 - H^n F^n(t)}{1 - HF(t)} + H^n (p^{(n)} * m)(t) \\ &\leq \frac{c}{1 - H} + H^n (p^{(n)} * m)(t) \end{aligned}$$

Now set  $m^*(t) = \sup_{0 \leq u \leq t} m(u)$ . Then via Lemma A1

$$\begin{aligned} (p^{(n)} * m)(t) &= \int_0^t p^{(n)}(t-u)m(u)du \\ &\leq m^*(t) \int_0^t p^{(n)}(u)du \leq m^*(t)F^n(t) \end{aligned}$$

Thus

$$\begin{aligned} m(t) &\leq \frac{c}{1 - H} + H^n F^n(t)m^*(t) \\ \Rightarrow m^*(t) &\leq \frac{c}{1 - H} + H^n F^n(t)m^*(t) \\ \Rightarrow m^*(t) &\leq \frac{c}{1 - H} \frac{1}{1 - H^n F^n(t)} \end{aligned}$$

Now let  $n \rightarrow \infty$  to get

$$m^*(t) \leq \frac{c}{1 - H} \Rightarrow m(t) \leq \frac{c}{1 - H}$$

as required. The proof is complete.

## B Calculation of $\Phi_l$

We derive a basic property using integration by parts:

$$\begin{aligned}
\Phi_l(t) &= \int_t^\infty \phi_l(u) du = \int_t^\infty \frac{(u\beta_o)^{l-1}}{(l-1)!} e^{-\beta_o u} \beta_o du \\
&= \int_{t\beta_o}^\infty \frac{v^{l-1}}{(l-1)!} e^{-v} dv \\
&= \frac{v^{l-1}}{(l-1)!} (-e^{-v}) \Big|_{t\beta_o}^\infty + \int_{t\beta_o}^\infty \frac{v^{l-2}}{(l-2)!} e^{-v} dv \\
&= \frac{(t\beta_o)^{l-1}}{(l-1)!} e^{-t\beta_o} + \Phi_{l-1}(t) \\
\Rightarrow \Phi_l(t) &= \frac{1}{\beta_o} \phi_l(t) + \Phi_{l-1}(t) \\
&= \Phi_{l-1}(t) - \frac{1}{\beta_o} \dot{\Phi}_l(t)
\end{aligned}$$

Integrating this gives

$$\int_0^T \Phi_l(t) dt = \mathcal{I}_l = \mathcal{I}_{l-1} + \frac{1}{\beta_o} (1 - \Phi_l(T)) \quad (2.1)$$

since  $\Phi_l(0) = 1$ . Also,  $\mathcal{I}_l(T) < \phi_l(0) = 1$  Note that  $\Phi_1(t) = e^{-\beta_o t}$  and so we have

$$\mathcal{I}_1 = \int_0^T e^{-\beta_o t} dt = \frac{1}{\beta_o} (1 - e^{-\beta_o T}) \quad (2.2)$$

The update for  $\Phi_l(T)$  is given by

$$\Phi_l(T) = \Phi_{l-1}(T) + \frac{1}{\beta_o} \phi_l(T) \quad (2.3)$$

with  $\Phi_1(T) = e^{-\beta_o T}$ .

## C Calculation of $B_l$ for HL

We take  $T_o = 0$  so that  $x_1(T_o) = x_1(0) = 0$ . Denote  $N_T = n$  so the final event time is  $T_n$ .

We now proceed step by step

### C.1 Calculation of $B_1$

We have

$$x_1(t) = \sum_{j: T_j < t} e^{-\beta_o(t-T_j)} \beta_o$$

$$\begin{aligned}
&= \sum_{j=1}^n H(t - T_j) e^{-\beta_o(t-T_j)} \beta_o \\
\Rightarrow B_1 &= \int_0^T x_1(t) dt \\
&= \sum_1^n \int_0^T H(t - T_j) e^{-\beta_o(t-T_j)} \beta_o \\
&= \sum_1^n \int_{T_j}^T e^{-\beta_o(t-T_j)} \beta_o \\
&= \sum_1^n \int_0^{T-T_j} e^{-\beta_o v} dv \\
&= \sum_1^n [-e^{-\beta_o v}]_0^{T-T_j} \\
&= \sum_1^n (1 - e^{-\beta_o(T-T_j)}) \\
&= n - \sum_1^n e^{-\beta_o(T-T_j)} = n - \frac{x_1(T)}{\beta_o}
\end{aligned}$$

### C.2 Calculation of $B_{l+1}$

For this case we have

$$\begin{aligned}
x_{l+1}(t) &= \sum_1^n H(t - T_j) \phi_l(t - T_j) \\
\Rightarrow B_{l+1} &= \int_0^T x_{l+1}(t) dt \\
&= \sum_1^n \int_{T_j}^T \phi_l(t - T_j) dt \\
&= \sum_1^n \int_0^{T-T_j} \phi_l(u) du \\
&= \sum_1^n (1 - \Phi_l(T - T_j)) \\
&= n - \sum_1^n \Phi_l(T - T_j)
\end{aligned}$$

Using the  $\Phi_l$  update from appendix A we get

$$\begin{aligned}
B_{l+1} &= n - \sum_1^n \Phi_{l-1}(T - T_j) - \frac{1}{\beta_o} \sum_1^n \phi_l(T - T_j) \\
&= B_l - \frac{1}{\beta_o} x_{l+1}(T)
\end{aligned}$$

## D Calculation of $\frac{d\lambda(u)}{d\beta_o}$

We need to calculate  $\frac{d\lambda(u)}{d\tau}$  where

$$\tau = \begin{bmatrix} \beta_o \\ \theta \end{bmatrix}, \theta = \begin{bmatrix} c \\ \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \text{ and}$$

$$\begin{aligned} \lambda(u) &= c + \sum_{l=1}^p \alpha_l \hat{x}_l(u) \\ \hat{x}_l(u) &= \hat{\lambda}_e \Phi_l(u) + x_l(u) \end{aligned} \quad (4.1)$$

So we need to calculate

$$\frac{d\lambda(u)}{d\tau} = \left[ \frac{d\lambda(u)}{d\beta_o}, \frac{d\lambda(u)}{d\theta} \right] \quad (4.2)$$

From 4.1, it is clear that

$$\frac{d\lambda(u)}{d\theta} = \begin{bmatrix} 1 \\ \hat{x}_1 \\ \dots \\ \hat{x}_p \end{bmatrix} \quad (4.3)$$

$$\frac{d\lambda(u)}{d\beta_o} = \sum_{l=1}^p \left( \alpha_l \frac{dx_l(u)}{d\beta_o} + \hat{\lambda}_e \frac{d\Phi_l(u)}{d\beta_o} \right) \quad (4.4)$$

We have:

$$\begin{aligned} x_l(t) &= \sum_{j:T_j < t} \phi_l(t - T_j) \\ \phi_l(u) &= \frac{e^{-u\beta_o} (\beta_o u)^{j-1} \beta_o}{(j-1)!} \end{aligned}$$

Thus

$$\begin{aligned} \frac{dx_l(t)}{d\beta_o} &= \frac{d}{d\beta_o} \left( \sum_{j:T_j < t} \phi_l(t - T_j) \right) \\ &= \sum_{j:T_j < t} \frac{d}{d\beta_o} \phi_l(t - T_j) \end{aligned}$$

Next

$$\begin{aligned} \frac{d\phi_l(u)}{d\beta_o} &= \frac{d}{d\beta_o} \left( e^{-\beta_o u} \frac{(\beta_o u)^{l-1} \beta_o}{(l-1)!} \right) \\ &= \frac{u^{l-1}}{(l-1)!} \left( e^{-\beta_o u} l \beta_o^{l-1} - u e^{-\beta_o u} \beta_o^l \right) \end{aligned}$$

$$\begin{aligned} &= \frac{u^{l-1}}{(l-1)!} \left( e^{-\beta_o u} l \beta_o^{l-1} \right) - \frac{u^{l-1}}{(l-1)!} u e^{-\beta_o u} \beta_o^l \\ &= e^{-\beta_o u} \frac{(\beta_o u)^{l-1}}{(l-1)!} \beta_o \left( \frac{l}{\beta_o} \right) - u \frac{u^{l-1}}{(l-1)!} e^{-\beta_o u} \beta_o^l \\ &= \phi_l(u) \frac{l}{\beta_o} - \frac{e^{-\beta_o u}}{(l-1)!} u^{l-1} \beta_o^{l-1} \beta_o u \\ &\Rightarrow \frac{d\phi_l(u)}{d\beta_o} = \phi_l(u) \frac{l}{\beta_o} - \phi_l(u) u \\ &= \frac{l}{\beta_o} (\phi_l(u) - \phi_{l+1}(u)) \end{aligned}$$

We have then

$$\begin{aligned} &\frac{dx_l(t)}{d\beta_o} \\ &= \sum_{j:T_j < t} \frac{d}{d\beta_o} \phi_l(t - T_j) \\ &= \sum_{j:T_j < t} \frac{l}{\beta_o} (\phi_l(t - T_j) - \phi_{l+1}(t - T_j)) \\ &= \frac{l}{\beta_o} (x_l(t) - x_{l+1}(t)) \end{aligned} \quad (4.5)$$

Continuing we have

$$\frac{d\Phi_l(t)}{d\beta_o} = \int_t^\infty \frac{d\phi_l(v)}{d\beta_o} dv$$

Using 4.5 we have,

$$\begin{aligned} &\frac{d\Phi_l(t)}{d\beta_o} \\ &= \int_t^\infty \left( \phi_l(v) \frac{l}{\beta_o} - \phi_l(v) v \right) dv \\ &= \frac{l}{\beta_o} \int_t^\infty \phi_l(v) dv - \int_t^\infty v \phi_l(v) dv \\ &= \frac{l}{\beta_o} \Phi_l(t) - \left( \frac{t}{\beta_o} \phi_l(t) + \frac{l}{\beta_o} \Phi_l(t) \right) \\ &= -\frac{t}{\beta_o} \phi_l(t) \end{aligned} \quad (4.6)$$

Placing 4.7, and 4.5, in 4.4, we get finally

$$\begin{aligned} \frac{d\lambda(t)}{d\beta_o} &= -\hat{\lambda}_e \sum_{l=1}^p \left( \frac{t}{\beta_o} \right) \alpha_l \phi_l(t) \\ &\quad + \sum_{l=1}^p \alpha_l \frac{l}{\beta_o} (x_l(t) - x_{l+1}(t)) \end{aligned}$$

## E Proof of Convergence in Result III

The convergence follows from lemma E2 below, whose proof requires lemma E1.

**Lemma E1.** If  $0 \leq K(v) \rightarrow K$  as  $v \rightarrow \infty$  then

$$\frac{1}{T} \int_0^T K(v)dv \rightarrow K \text{ and } \frac{1}{T} \int_0^T |K(v) - K|dv \rightarrow 0$$

*Proof.* The second result implies the first. Given  $\epsilon > 0$  we can find  $T_\epsilon$  so that for all  $v \geq T_\epsilon$  we have  $|K(v) - K| < \epsilon$ . Then

$$\begin{aligned} d_T &= \frac{1}{T} \int_0^T |K(v) - K|dv \\ &= \frac{1}{T} \int_0^{T_\epsilon} |K(v) - K|dv + \frac{1}{T} \int_{T_\epsilon}^T |K(v) - K|dv \end{aligned}$$

Now 1st term  $\rightarrow 0$  as  $T \rightarrow \infty$  and  $|2nd - term| \leq \epsilon$ . Thus  $\limsup d_T \leq \epsilon$ . But  $\epsilon$  is arbitrary and the result follows.

**Lemma E2.** Suppose  $0 \leq K(v) \rightarrow K$  as  $v \rightarrow \infty$  then

$$\frac{1}{T} \int_0^T K(v)m(T-v)dv \rightarrow K\lambda_e$$

*Proof.* We use lemma E1 repeatedly. Since  $m(t) \rightarrow \lambda_e$  then

$$\frac{K}{T} \int_0^T m(T-v)dv = \frac{K}{T} \int_0^T m(u)du \rightarrow K\lambda_e$$

So we need only show

$$d_T = \frac{1}{T} \int_0^T [K(v) - K]m(T-v)dv \rightarrow 0$$

Again since  $\frac{\lambda_e}{T} \int_0^T (K(v) - K)dv \rightarrow 0$  we need only show

$$d_T = \frac{1}{T} \int_0^T (K(v) - K)(m(T-v) - \lambda_e)dv \rightarrow 0$$

Result II gives  $0 \leq m(u) \leq B < \infty$  for all  $u$ . Thus

$$\begin{aligned} |d_T| &\leq \sup_{0 \leq v \leq T} |m(T-v) - \lambda_e| \frac{1}{T} \int_0^T |K(v) - K|dv \\ &\leq (B + \lambda_e) \frac{1}{T} \int_0^T |K(v) - K|dv \rightarrow 0 \end{aligned}$$

and the lemma is established.

## F Calculation of $B_l$ for BAR

From the BAR subsection we have

$$\delta x_l(t) = N_{t-l\delta+\delta} - N_{t-l\delta} = \int_{t-l\delta}^{t-l\delta+\delta} dN_u$$

By inspection we make the crucial observation that

$$x_l(t) = 0 \text{ for } t \leq l\delta - \delta$$

Thus

$$\begin{aligned} \delta B_l &= \int_0^T \delta x_l(t)dt = \int_{l\delta-\delta}^T \delta x_l(t)dt \\ &= \int_{l\delta-\delta}^T \left( \int_{t-l\delta}^{t-l\delta+\delta} dN_u \right) dt \end{aligned}$$

Now change variables to  $\tau = t - l\delta + \delta$  to get

$$\begin{aligned} \delta B_l &= \int_0^{T-l\delta+\delta} \left( \int_{\tau-\delta}^{\tau} dN_u \right) d\tau \\ &= \int_0^{T-l\delta+\delta} \left( \int_0^{\tau} dN_u - \int_0^{\tau-\delta} dN_u \right) d\tau = a_l - b_l \end{aligned}$$

Continuing

$$a_l = \int_0^{T-l\delta+\delta} \left( \int_0^{\tau} dN_u \right) d\tau$$

Change integration order to get

$$\begin{aligned} a_l &= \int_0^{T-l\delta+\delta} dN_u \left( \int_u^{T-l\delta+\delta} d\tau \right) \\ &= \int_0^{T-l\delta+\delta} dN_u (T - l\delta + \delta - u) \\ &= \sum_{T_i < T-l\delta+\delta} (T - l\delta + \delta - T_i) \end{aligned}$$

Next

$$b_l = \int_0^{T-l\delta+\delta} \left( \int_0^{\tau-\delta} dN_u \right) d\tau$$

Change the order of integration to get

$$b_l = \int_0^{T-l\delta+\delta} dN_u \left( \int_{u+\delta}^{T-l\delta+\delta} d\tau \right)$$

$$\begin{aligned}
&= \int_0^{T-l\delta+\delta} dN_u(T-l\delta-u) \\
&= \Sigma_{T_i < T-l\delta+\delta} (T-l\delta-T_i) \\
&= \Sigma_{T_i < T-l\delta+\delta} (T-l\delta+\delta-T_i-\delta) = a_l - d_l
\end{aligned}$$

where

$$d_l = \delta \Sigma_{T_i < T-l\delta+\delta} 1 = \delta N_{T-l\delta+\delta}$$

Thus we find

$$\delta B_l = a_l - b_l = a_l - [a_l - d_l] = d_l = \delta N_{T-l\delta+\delta}$$

and the claimed result follows.

## G Proof of Result IV

We first calculate  $\gamma_+(0)$ . From the remark following result IV we get

$$\begin{aligned}
\gamma_+(0) &= \lambda_e \int_{-\infty}^{\infty} [\bar{g}(j\omega) + \bar{g}(-j\omega) + |\bar{g}(j\omega)|^2] \frac{d\omega}{2\pi} \\
&= 2\lambda_e g(0) + \lambda_e \int_{-\infty}^{\infty} |\bar{g}(j\omega)|^2 \frac{d\omega}{2\pi}
\end{aligned}$$

To continue we need an expression for  $\bar{g}(s)$ . From (??) we find

$$\begin{aligned}
\bar{h}(s) &= \sum_1^p \alpha_l \int_0^{\infty} e^{-su} \frac{(\beta_o u)^{l-1}}{(l-1)!} e^{-\beta_o u} \beta_o du \\
&= \sum_1^p \alpha_l \frac{\beta_o^l}{(\beta_o + s)^l} \int_0^{\infty} \frac{[(\beta_o + s)u]^{l-1}}{(l-1)!} e^{-(\beta_o + s)u} du (\beta_o + s) \\
&= \sum_1^p \alpha_l \frac{1}{(\tau_o s + 1)^l} \int_0^{\infty} \frac{x^{l-1}}{(l-1)!} e^{-x} dx \\
&= \sum_1^p \alpha_l \frac{1}{(\tau_o s + 1)^l} = \frac{\alpha(\xi)}{\xi^p}
\end{aligned}$$

where  $\xi = \tau_o s + 1$  and  $\alpha(\xi) = \sum_1^p \alpha_l \xi^{p-l}$ . It follows that

$$\bar{g}(s) = \frac{\alpha(\xi)/\xi^p}{1 - \alpha(\xi)/\xi^p} = \frac{\alpha(\xi)}{\xi^p - \alpha(\xi)}$$

Note that  $\bar{g}(s)$  is strictly proper and in view of result IV is stable. Using the Laplace transform initial value theorem we find

$$g(0) = \lim_{s \rightarrow \infty} s \bar{g}(s)$$

Next

$$\begin{aligned}
&= \lim_{s \rightarrow \infty} s \frac{\sum_1^p \alpha_l (\tau_o s + 1)^{p-l}}{(\tau_o s + 1)^p - \sum_1^p \alpha_l (\tau_o s + 1)^{p-l}} \\
&= \lim_{s \rightarrow \infty} s \frac{\sum_1^p \alpha_l / (\tau_o s + 1)^l}{1 - \sum_1^p \alpha_l / (\tau_o s + 1)^l} = \alpha_1 / \tau_o \\
V_g &= \int_{-\infty}^{\infty} \frac{|\bar{g}(j\omega)|^2 \frac{d\omega}{2\pi}}{|(\tau_o j\omega + 1)^p - \alpha(\tau_o j\omega + 1)|^2} \frac{d\omega}{2\pi} \\
&= V_g / \tau_o \\
V_g &= \int_{-\infty}^{\infty} \frac{|\alpha(j\omega' + 1)|^2 \frac{d\omega'}{2\pi}}{|(j\omega' + 1)^p - \alpha(j\omega' + 1)|^2} \frac{d\omega'}{2\pi}
\end{aligned}$$

Thus we find

$$\gamma_+(0) = \frac{\lambda_e}{\tau_o} \mathcal{J} \text{ where } \mathcal{J} = 2\alpha_1 + V_g$$

where  $\mathcal{M}, \mathcal{J}$  do not depend on  $\tau_o$  but only on memory parameters.

To complete the proof we derive the lower bound.

Using Parseval's theorem, result IV(iv) and the HL-HIR bound we find

$$\begin{aligned}
V_g = \tau_o \int_0^{\infty} g^2(t) dt &\leq \tau_o \frac{B_h H}{(1-H)^2} \\
&= \tau_o \beta_o \frac{H}{(1-H)^2} = \frac{H}{(1-H)^2}
\end{aligned}$$

Thus we find

$$\begin{aligned}
\mathcal{J} &\leq \alpha_1 + \frac{H}{(1-H)^2} \\
\Rightarrow L &\geq \tau_o \frac{H(2-H)}{(1-H)^2} \frac{1}{[2\alpha_1 + H/(1-H)^2]} \\
&= \tau_o \frac{H(2-H)}{H + 2\alpha_1(1-H)^2}
\end{aligned}$$